Mathematical Models of Beams and Cables

Mathematical Models of Beams and Cables

Angelo Luongo
Daniele Zulli

Series Editor
Noël Challamel

ISTE

WILEY

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

First published 2013 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

Apart from any fair dealing for the purposes of research or private study, or criticism or review, as permitted under the Copyright, Designs and Patents Act 1988, this publication may only be reproduced, stored or transmitted, in any form or by any means, with the prior permission in writing of the publishers, or in the case of reprographic reproduction in accordance with the terms and licenses issued by the CLA. Enquiries concerning reproduction outside these terms should be sent to the publishers at the undermentioned address:

ISTE Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

www.wiley.com

© ISTE Ltd 2013

The rights of Angelo Luongo and Daniele Zulli to be identified as the author of this work have been asserted by them in accordance with the Copyright, Designs and Patents Act 1988.

Library of Congress Control Number: 2013948060

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN: 978-1-84821-421-7

MIX
Paper from responsible sources
FSC® C013604
FSC www.fsc.org

Printed and bound in Great Britain by CPI Group (UK) Ltd., Croydon, Surrey CR0 4YY

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Table of Contents

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Preface

It is customary, in the engineering community, to distinguish linear from nonlinear theories. As a matter of fact, “Dynamics” and “Nonlinear Dynamics”, as well as “Buckling” and “Post-Buckling” are considered to be basically different disciplines. Usually, the adjective “linear” is omitted so that theories, “by default”, are understood to be linear, and nonlinearities, when accounted for, must be explicitly mentioned. This habit is probably due to the fact that, usually, linear models are believed sufficient to solve most of the technical problems encountered in practice. Linear theories, therefore, are well rooted in the knowledge of any engineer, whereas nonlinear theories are considered to be of the competence of few specialists.

The opposite, instead, occurs in the field of Applied Mathematics. For example, “Continuum Mechanics” and “Dynamical System Theory” are understood as intrinsically nonlinear, with no need for further specifications. If nonlinearities are instead ignored, mathematicians stress that the problem has been linearized. This practice is probably due to the fact that mathematicians are not very interested in solving technical problems and, therefore, they are not stimulated by “simplifying” models, but rather by formulating theories and proving theorems of the widest generality.

The theory of beams, which is the subject of this book, is not an exception to this rule. The relevant linear theory is well-known to mechanical and civil engineers, but only some researchers and few PhD students operating in the field are confident with nonlinear theories. Thus, if an engineer tries to enter this new world, he or she must face the reading of books whose mathematics is often beyond his or her knowledge; the focus is on the formulation in itself, and often little (or even no) attention is paid to algorithms and examples. In other words, the engineer has to face an environment where mathematics “is a end, but not a means”. On the other hand, there exist some good books specialized in one of the subjects discussed above. However, very often, they are entirely devoted to illustrating algorithms and results, while modeling is overlooked or required as a reader’s prerequisite.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

All previous considerations seemed to be good reasons for the authors to write a new book, which is aimed (a) at introducing the reader to geometrically exact nonlinear modeling of one-dimensional structures, by using elementary mathematics only; (b) at consistently deriving approximate models in order to render the relevant equations amenable to semi-analytical solutions; and (c) at giving a comprehensive overview of different engineering problems, in which the application of nonlinear theories is mandatory. However, after having written a few chapters, the project soon appeared too ambitious to be fully realized in a single volume. Therefore, it was decided to devote a first book to models and a forthcoming book (*Nonlinear Beam and Cable Mechanics in Engineering Applications*) to algorithms and phenomena, with the aim of guiding the reader throughout the whole process of the engineering design. The two books, therefore, should be intended as sequential and closely related, but, at the same time, independent, so that only one of the two aspects, theoretical or algorithmic-phenomenological, can be addressed by the readers, consistently with their interest.

In this book, several models of elastic and viscoelastic beams, both in statics and dynamics, are analyzed. They are discussed in order of increasing complexity by including straight/curved, planar/non-planar, extensible/inextensible, shearable/unshearable, warpable/unwarpable, cross-deformable/undeformable and prestressed/unprestressed beams. String and cables, straight or curved, perfectly flexible or endowed with flexural–torsional stiffnesses, are also addressed. Modeling is developed via a direct approach, based on one-dimensional polar or Cauchy continua.

In summary, this book is an attempt (a) to make it easy to learn the nonlinear theory of beams and cables and (b) to formulate consistent approximate models, leading to reasonably simple mathematical problems.

The book is mainly devoted to researchers and PhD students in Civil and Mechanical Engineering, as well as in Applied Mathematics. It is also hoped to be useful for professional engineers. It requires only the basic knowledge of Mathematical Analysis, Linear Algebra and Continuum Mechanics, generally covered in engineering courses.

Angelo Luongo
Daniele Zulli
September 2013

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Introduction

Here we summarize the main concepts to be discussed later and illustrate the guiding factor of this book. Firstly, the modeling problem for a beam is addressed by comparing two different philosophies: derivation from a three-dimensional (3D) Cauchy continuum, or direct formulation as a one-dimensional (1D) polar continuum. Secondly, string and cables are successively considered as degenerate models of perfectly flexible beams, and the circumstances in which flexural and torsional stiffnesses have to be considered are discussed. Thirdly, more sophisticated models of beams with deformable cross-sections are addressed. Finally, a quick overview of the literature and of this book is given.

I.1 Derived one-dimensional models

A beam is a slender solid, spanned by a planar figure $\mathcal{A}$ (the *cross-section*) which moves along a smooth (C^1 class) curve $\mathcal{S}$ (the *beam axis* or centerline), by remaining orthogonal to it and keeping its centroid G on it. If $\mathcal{S}$ is a planar curve, the beam is called *planar*, otherwise it is *spatial*; if $\mathcal{S}$ is a segment, then the body is a cylinder, and the beam is called *straight*. The length l of $\mathcal{S}$ is called the *beam length.* Slenderness, in a broad sense, refers to the fact that a characteristic linear dimension r of the cross-section $\mathcal{A}$ is much less than the length l (typically $l/r = \mathrm{O}(10^2)$). This property plays a fundamental role in deriving mechanical models of beams, as discussed further.

As is well-known, the *Fundamental Problem of Continuum Mechanics*, formulated in the context of the Lagrangian description, consists of evaluating *stresses, strains and displacements in a body*, when this is loaded by assigned volume and surface forces, and, moreover, displacements are prescribed on a portion of the boundary. When this problem is addressed for a beam, a 3D Cauchy continuum model could be applied, which would lead to a system of partial differential equations, in which each (scalar, vector or tensor) magnitude is a function of three coordinates (time understood), which span the volume occupied by the body

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

in the reference configuration. Such an approach, however, although “exact” in the context of continuum mechanics, is almost unpractical for the difficulty of solving the governing equations, so that it is advisable to resort to “approximate” models that exploit the geometric peculiarity of the body, namely its slenderness. The main object of the analysis consists of formulating a 1D (rather than a 3D) model in which all the magnitudes involved depend on only one coordinate, e.g. a curvilinear abscissa s running along the (unstretched) curve $\mathcal{S}$.

To achieve this goal, different methods can be followed. We first approach the problem by illustrating how to derive a 1D model from a 3D model. Let us consider the displacement $\mathbf{u}(\mathbf{r}, s)$ at a point P belonging to the section $\mathcal{A}$ at abscissa s, where $\mathbf{r} := \overrightarrow{PG}$ is the oriented distance of P from the centroid G of $\mathcal{A}$. By using a Taylor expansion, it follows that $\mathbf{u}(\mathbf{r}, s) = \mathbf{u}(\mathbf{0}, s) + \partial_{\mathbf{r}}\mathbf{u}(\mathbf{0}, s)\mathbf{r} + \mathrm{O}\left(\|\mathbf{r}\|^2\right)$. Since $\|\mathbf{r}\| \ll l$, it can be assumed that the remainder of the series is negligible, with an error which, in non-dimensional variables, is of the order of $(r/l)^2$. As a result, the displacement at *any* point of the section is expressed as a function of quantities *all* evaluated at the centroid, namely $\mathbf{u}(\mathbf{r}, s) = \mathbf{u}_G(s) + \partial_{\mathbf{r}}\mathbf{u}_G(s)\mathbf{r}$ with $\mathbf{u}_G(s) := \mathbf{u}(\mathbf{0}, s)$ and $\partial_{\mathbf{r}}\mathbf{u}_G(s) := \partial_{\mathbf{r}}\mathbf{u}(\mathbf{0}, s)$. Once this displacement field is introduced in a variational principle (e.g. if the beam is elastic, in the total potential energy principle), ordinary (instead of partial) differential equations are obtained, the unknowns now only depend on the coordinate s. The approximation, of course, could be improved by retaining higher-order terms up to a desired one, this entailing the need for introducing higher-order derivatives. These, however, do not change the essence of the Taylor expansion, which consists of extrapolating information from the beam axis to points external to the axis, but “close” to it, when one looks at the beam on a scale of characteristic length l.

The previous kinematic description, when truncated at the first order, expresses the displacement field on the section as linearly dependent on the distance $\mathbf{r}$ from the centroid (note that this property has nothing to do with the magnitude of displacements that, indeed, can be arbitrarily large, but is only a consequence of the “closeness” of P to G). The field depends on nine scalar parameters, three contained in the vector $\mathbf{u}_G(s)$, describing translations of the section, and six in the 3×2 matrix of $\partial_{\mathbf{r}}\mathbf{u}_G(s)$. The latter, suitably combined among them (generally in a nonlinear way), are responsible for a rotation (three parameters) and an affine transformation of the cross-section in its own plane (other three parameters). If we neglect the latter, by assuming that *the section is rigid in its own plane*, we reduce the motion to $\mathbf{u}(\mathbf{r}, s) = \mathbf{u}_G(s) + (\mathbf{R}(s) - \mathbf{I})\mathbf{r}$, where $\mathbf{I}$ is the identity tensor, and $\mathbf{R}(s)$ is the rotation tensor, (nonlinearly) depending on three scalar parameters θ_1, θ_2 and θ_3. This result suggests we look at the beam from a different perspective, namely as a collection of infinitely many rigid bodies of evanescent thickness (the cross-sections), supported by a flexible line (the beam axis). Each section, therefore, has six degrees of freedom, being only capable of translating and rotating, not deforming itself. Of

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

course, this is quite a rough model because we know, even from the linear de Saint-Venant Theory, that the cross-sections do deform in their own plane, and, moreover, they warp out-of-plane. However, we also know that the former strains represent less important aspects of kinematics, whereas the latter effects are very important only in thin-walled beams, which therefore call for a special treatment. On the other hand, renouncing the description of some aspect of the problem represents a compromise between the desired accuracy and the complexity of the equations to be solved.

I.2 Direct one-dimensional models

In the former approach, the 1D model was derived from a 3D one, by constraining, in some sense, the kinematics. However, we could wonder if it is possible to formulate a "direct" 1D model of a beam, by avoiding "derivation" from a more complex system. This, indeed, is possible, by referring, however, to a continuum richer than Cauchy's, namely to a *polar continuum* (also known as Cosserat's continuum or a structured continuum). A polar continuum is made of material particles that are endowed with orientation; they, therefore, can translate, as the particles of the Cauchy continuum, but, in addition, can rotate. (This rotation is often called the *micro-rotation*, to distinguish it from the macro-rotation that one observes in a Cauchy continuum, when one looks not at a particle, but at a small neighborhood of it). With this idea in mind, we can consider the beam as a 1D object, geometrically described by its axis $\mathcal{S}$, which, however, is made up of orientable material particles, in other words, as a 1D polar continuum. Each point of this continuum possesses six degrees of freedom, as the 3D beam made up of rigid sections. If we compare the two points of view (3D and 1D objects), we observe that the cross-sections $\mathcal{A}$ disappear in the latter model; however, they are replaced by body-points, able to describe, via their orientation, the cross-section attitude, thus regaining the information lost. (Of course, if the cross-section was also able to deform itself, in addition to rotating, more information should be borne by the structured continuum, as we will discuss later).

How can we account for the orientation of body-points? The best way is to consider that a triad of orthonormal unit vectors $\mathbf{a}_i(s)$, called *directors*, is attached to each of them. The rotation of the triad, described by a rotation tensor $\mathbf{R}(s)$, describes the rotation of the body-point. Although not strictly necessary, one can think that two of the directors lie, in any configuration, in the plane of the (now disappeared) cross-section, whereas the third one, orthogonal to former ones, is initially aligned with the beam axis, but, after the deformation, loses this parallelism.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

It is interesting to compare derived and direct models not only from a kinematic point of view, but also from a dynamical perspective. In a Cauchy 3D continuum, in-contact points only exchange forces per unit area; in a 1D polar continuum, instead, they exchange forces and couples. When we consider a beam as a 3D object, we have to integrate the stresses on the cross-section, to evaluate their resultants, namely the axial and shear forces, and the bending and torsional moments. When, instead, we use the 1D polar continuum, the internal forces already represent these resultants. Therefore, the absence of an "internal arm", which is responsible for the moment in the 3D model, is regained, in the 1D polar continuum, by the dynamic property of the body-points to exchange couples.

Using a direct model, instead of deriving it from a 3D continuum, offers some advantages. Indeed, the direct model is capable to describe *beam-like structures*, as trussed or framed beams (Vierendel-like), or, moreover, spiral structures (as helicoidal springs), when one is not interested in the local behavior (e.g. of the single truss), but rather in the global behavior of the structure as a whole. These systems put in evidence a new question: how can we endow the model with a constitutive law? The problem, however, is not specific to beam-like structures, but also concerns massive beams and represents, in some sense, the drawback of direct models. If, indeed, in the derived models, the constitutive law is a straightforward consequence of that of the Cauchy continuum, in contrast, in the direct model, the law has to be independently enforced. The problem can be tackled by considering a *representative volume* of the refined model (e.g. one period length of a periodic structure) and establishing an energy equivalence between this and an equally long segment of the rough model.

In this book, we follow the direct approach and will show how to formulate constitutive laws for massive and beam-like structures.

I.3 Cables and strings

When the beam possesses a very high slenderness ratio (e.g. of the order of 10^3, or more), it becomes extremely flexible. For example, if it is disposed (without stretch) on a horizontal line, hinged at the ends, and then made subject to its own weight, it undergoes large, prevalently transverse, displacements, thus resulting as axially stressed rather than subject to a bending moment. Such an object, of course, would be of no utility in structural engineering, where beams are designed to carry loads. However, if the beam is put into a state of tensile prestress, it can assume a significant *geometric stiffness*, which makes the structure work. As is well-known from the elementary theory of beams, the geometric stiffness is produced by the perturbation of the pre-existent state of equilibrium, which brings unbalanced forces up, able to balance the incremental loads. Such a mechanical system is usually not

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

recognized as a beam, but rather as a *string* or *cable*, the first denomination being preferred when the axis is almost rectilinear (i.e. when the prestress is mainly due to tensile end forces), and the second when the axis is significantly curved (or piecewise linear, i.e. when the prestress is due to carried weights).

The simplest (and nearly universally used) model for a string completely neglects the flexural (and torsional) stiffness. Therefore, a string is viewed as a perfectly flexible, idealized beam, whose cross-sectional area is lumped at the centroid, and, consequently, has zero moment of inertia. More precisely, we can also say that a string is a 1D Cauchy continuum (instead of polar), embedded in a 3D environment. As a result of this idealization, the string does not possess its own specific shape, since it can assume infinite natural configurations in which stress and strain simultaneously vanish. In the linear field, therefore, the string is a *kinematically undetermined and statically impossible continuum system*; prestress, however, makes the equilibrium possible in a configuration very close to the prestressed one (often called *adjacent configuration*), as occurs, for example, for a mathematical pendulum prestressed by the gravity force, when it is disturbed by a transverse force. The prestressed configuration is usually taken as reference configuration in a Lagrangian approach.

There exist, however, some problems in which the perfectly flexible model is not adequate to give an answer. We cite two of these circumstances. (a) If the string is clamped at the ends, a *boundary layer* manifests itself in a narrow zone close to the clamps, in which the bending effects cannot be neglected. The same occurs close to transverse loads applied to the string, as those transmitted by a pulley moving along the cable. Without going into detail, which is out of the scope of this book, it can be checked that the flexural stiffness appears as a small term affecting the highest derivatives in the equations of motions. This term can be neglected almost everywhere (to obtain the so-called *outer solution*), but not close to singular points (where an *inner solution* must be sought). Here, in order to satisfy the boundary conditions, the solution becomes fast varying, thus rendering the (elsewhere) small terms comparable to the leading terms. If, therefore, one uses the perfectly flexible model, one has to renounce satisfying some of the boundary conditions and, consequently, investigating the mechanical state in the boundary layers. (b) As a second example, let us consider the effect of wind on iced strings. Ice accretion modifies the (usually) original circular shape of the section and consequently its aerodynamic properties. Therefore, wind loads depend on the attitude of the cross-section, and change during motion, as a consequence of the twisting of the string. To correctly analyze this interaction phenomenon, torsional effects need to be introduced into the structural model.

One could, of course, use a complete model of a prestressed beam to analyze all problems involving strings and cables, without introducing the drastic simplification concerning their flexibility. This is usually done in purely numerical investigations,

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where all the terms of the beam model are retained in the analysis, and used together with prestress. However, such models are complex and suffer from ill-conditioning, since the relevant stiffness matrices are nearly singular, for the presence of small terms. An approximated simple model, accounting for the essential terms, is discussed later in this book.

I.4 Locally deformable beams

As we observed before, there are problems in which deformations of the cross-section cannot be ignored. The question assumes great importance when the beam is thin-walled, open or closed. As a first example, it is known from the Vlasov theory [VLA 61] that warping of open thin-walled beams, when caused by non-uniform torsion, induces stresses normal to the cross-section (equivalent to a *bi-moment*), variable along the beam-axis; these, in turn, trigger tangential stresses equivalent to a (so-called secondary) torsional moment which cannot be ignored. Consequently, the torsional stiffness of the beam turns out to be much higher than the de Saint-Venant stiffness. As a second example, it is known from the Brazier theory [BRA 27] that when a tubular beam is bent, the original circular middle line undergoes ovalization, with flattening and consequent reduction of the cross-section inertia moment.

A proper modeling of a thin-walled beam therefore calls for accounting for in-plane and/or out-of-plane deformability of the cross-sections. We will refer to these beams as *locally deformable* (and to the former as locally undeformable). The task can, again, be accomplished via derivation from a 3D model or via a direct approach, as here briefly outlined.

The modern generalized beam theory (GBT, see e.g. [GON 07, BEB 08, BAS 09, SIL 10, CAM 10, GON 10] for a wide overview) derives a 1D model from the 3D Cauchy continuum. It is based on the semi-variational (or Kantorovitch) method, according to which the displacements $\mathbf{u}(\mathbf{r}, s)$ are expressed as a linear combination of *known shape functions* $\boldsymbol{\psi}(\mathbf{r})$ and *unknown amplitude functions* $\boldsymbol{a}(s)$. A variation principle leads to a set of ordinary differential equations in the amplitudes. Of course, if the shape functions only describe rigid motions of the cross-section, the GBT furnishes the locally undeformable model; for this reason, it is called *generalized*, since it includes the standard model. However, it is much more powerful, because it is capable of accounting for changes of shape of the cross-section, including warping.

If a direct model is desired, the classical Cosserat continuum must be endowed with *additional kinematic descriptors* $\boldsymbol{a}(s)$, here called *distortional variables*, whose meaning, at least initially, is not necessary to be specified. These descriptors entail that the strains of the beam increase in number, with respect to the standard model.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

In the context of a first-gradient theory, strains consist of the descriptors themselves, $\boldsymbol{\alpha}(s) := \boldsymbol{a}(s)$ and of their first derivatives $\boldsymbol{\beta}(s) := \boldsymbol{a}'(s)$. The use of a variational principle leads to balance equations in the displacements and distortional variables. The main difficulty of this approach consists of assigning the constitutive law, which links the generalized strains to their dual generalized stresses (i.e. *distortional* and *bi-distortional* stresses). This task can be accomplished by an identification procedure from a 3D model, based on an equivalence in energy. The operation leads to attributing a geometrical meaning to the distortional variables and a mechanical meaning to the dual stress quantities.

In this book, we will follow the direct approach and use a 3D fiber-model to identify the constitutive law.

I.5 An overview of literature

The literature on beams and cables is extensive and continuously produced over the years. A huge quantity of books and scientific papers are issued daily on the topics, and it would be in any case impossible to try to overview most of them. Therefore, we limit ourselves, here, to cite only some classical and recent books, considering that the literature on beams and cables should be necessarily embedded in the wider context of continuum mechanics, rational mechanics, linear and nonlinear dynamics and stability theory.

From this point of view, the main reference is given to classic benchmarks on continuum mechanics [GUR 82, GUR 72, GUR 83, GUR 00, TRU 77, TRU 66, TRU 04, CIA 88, VIL 77, GRE 92, LAN 70, LEI 74, MAL 69, MAR 93, OGD 07, POD 00, SOU 73, TIM 51, FUN 01, HOL 00, DEN 87, WAS 82, ODE 82, RED 02] as well as to more recent contributions [GUR 10, OGD 97, CHA 13, CON 07, WEG 09, DIM 11, DYM 13, ROM 06, BER 09, BAR 10, BRI 13, ERE 13, ESL 13, RED 10, RED 13].

Fundamental concepts about the mathematical framework, general mechanics, stability theory and nonlinear dynamics can be found in [COU 53, CHO 01, CLO 03, FIN 08, GAL 07, GAN 13, GOL 80, GRE 10, KOK 06, KUI 99, LAN 97, LEI 87, LOV 89, MAR 12, MEI 70, MEI 97, MEI 01, MEI 80, NAY 79, NAY 73, WEI 74, PRE 13].

Many books move freely from the general continuum mechanics to more specific theories of beams, plates or shells. Among them, reference is made to [ANT 05, LOV 44, NAY 04, BIG 12, HOW 09, LAC 13, RUB 00]. Sometimes the subject is approached in the finite element context, as in [BAT 82, WRI 08, ZIE 05, IBR 09].

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

On the other hand, books specifically devoted to cables, beams and/or shells are [ANT 72, CAP 89, VIL 97, TIM 65, TIM 59, VLA 61, FER 06, LIB 06, IRV 81, MAG 12, MUR 86, OBO 13, ROS 11, VIN 89, VOR 99, WAN 00, HOD 06, HOD 11] and those devoted to their stability are [BOL 64, BOL 63, ATA 97, BAŽ 03, ELI 01, LEI 87, PIG 92, SIM 06, AMA 08, TIM 63].

As a specific choice of the authors, only a few journal papers are reported here, the literature overview being more turned towards books. Nevertheless, we find it important to cite here some fundamental contributions reported in papers: on general framework and continuum mechanics [GER 73, DIC 96], on cables [IRV 74, REG 04a, REG 04b, IBR 04, PER 87, LU 94, LEE 92, BUR 88, TJA 98, TRI 84, GAT 02, GOY 07, GOY 05], on beams [SIM 85, SIM 86, SIM 88, SIM 91, CRE 91, CRE 78a, CRE 78b, ZAR 94, CRE 88a, CRE 88b], on thin-walled beams [BRA 27, RIZ 96, DIC 99, REI 59, REI 83a, REI 83b, REI 84, REI 87, RUT 06], and on beam-like structures [DIC 90] and GBT [SIL 03, GON 07, BAS 09, BEB 08, CAM 06, SIL 10, CAM 10, GON 10].

I.6 An overview of the book

All the concepts discussed above are detailed in this book. Here, a short overview is presented. The book is ideally divided into four parts: (a) metamodel (Chapter 1), (b) locally undeformable beams (Chapters 2–4), (c) cables and strings (Chapters 5 and 6), and (d) locally deformable thin-walled beams (Chapters 7 and 8).

In Chapter 1 a metamodel is introduced, which works as a progenitor for specific models to be dealt with later. Here, unprestressed, prestressed and internally constrained beams, with or without prestress, are studied. The virtual power principle is used to derive the balance equations [GER 73]. The variational formulation is also illustrated. Exact equations are derived in operator form and then linearized around the reference configuration.

Planar straight beams, locally undeformable, are addressed in Chapter 2. Exact kinematics and balance equations are derived. Homogenization procedures are outlined to derive the constitutive law. The model is developed in a 3D environment. The planar beam is drawn as a particular case.

The previous analysis is extended in Chapter 3 to curved beams (arches), both in-space and in-plane.

Chapter 4 deals with internally constrained beams for which two basically different approaches (mixed and displacement formulations) are followed, respectively, including or not the reactive stresses produced by the internal

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

constraints. Several cases of internally constrained beams are considered; a few among them are unshearable, inextensible, untwistable, and shear–shear–torsional beams.

In Chapter 5 flexible cables and strings are analyzed as 1D bodies *not* endowed with flexural and/or torsional stiffnesses. Both unprestressed and prestressed cables are considered and linearized equations are derived for the latter group. Approximated equations for shallow cables, horizontal or inclined, are obtained. Finally, inextensible cables are addressed.

In Chapter 6 stiff cables, equipped with flexural and torsional stiffnesses, are considered. A simplified model, based on the hypothesis of small curvature and large elongation, is developed.

A 1D model of a thin-walled beam undergoing in-plane and out-of-plane distortions of the cross-section is formulated in Chapter 7. Nonlinear hyper-elastic laws are obtained by the homogenization process of a 3D fiber-model. Due to their cumbersome expressions, governing equations are explicitly given only for simple cases, although the illustrated procedure is general.

The theory is specialized in Chapter 8 to locally deformable thin-walled beams with internal constraints, of the Vlasov, Bredt and Brazier type, able to supply nonlinear equations which generalize the underlying linear theories, respectively.

I.7 Notation

Throughout the book, a scalar quantity is denoted by a Roman or Greek italic letter (e.g. u or ω). A bold Roman or Greek letter denotes a vector or a tensor: the former (mostly, but not exclusively) lowercase (e.g. $\mathbf{u}$ or $\boldsymbol{\omega}$) and the latter (exclusively) uppercase (e.g. $\mathbf{R}$). A bold italic letter refers to a column-matrix or matrix (e.g. $\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{R}$) which, typically, are the scalar representation of the homonymous vector quantities in a specified basis, as discussed below.

Vectors and tensors attached to bases

We will often handle vectors and tensors *attached* to bases. For example the vector $\bar{\mathbf{r}} := \overrightarrow{OP}$, denoting the relative position of the points P and O of the space attached to the basis $\bar{\mathcal{B}}$, is itself an attached vector. If the space undergoes a rotation $\mathbf{R}$ (remember that $\mathbf{R}^{-1} = \mathbf{R}^T$) which brings the basis $\bar{\mathcal{B}}$ to another basis $\mathcal{B}$, then $\bar{\mathbf{r}}$ rotates with the space, and changes into a *new* vector $\mathbf{r} := \mathbf{R}\bar{\mathbf{r}}$. Consequently, if $\bar{\mathbf{L}}$ is a tensor which transforms the vector $\bar{\mathbf{u}}$ into $\bar{\mathbf{v}}$, both attached to $\bar{\mathcal{B}}$, (i.e. $\bar{\mathbf{v}} = \bar{\mathbf{L}}\bar{\mathbf{u}}$), then $\mathbf{L} := \mathbf{R}\bar{\mathbf{L}}\mathbf{R}^T$ is

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

a *new* tensor which transforms the rotated vector $\mathbf{u} := \mathbf{R}\bar{\mathbf{u}}$ into $\mathbf{v} = \mathbf{R}\bar{\mathbf{v}}$ (i.e. $\mathbf{v} = \mathbf{L}\mathbf{u}$). Note that, by taking $\mathbf{L} = \mathbf{R}$, it follows, that $\bar{\mathbf{R}} = \mathbf{R}$.

Scalar representation of vectors and tensors in different bases

The aforementioned transformations describe the change, under a rotation, of *absolute geometric entities into new geometric enties*. As is well-known to the reader, they should not be confused with the transformations undergone *by the components* of a (sole) geometric entity when the basis is rotated. To relate the components of a given vector $\mathbf{w}$ or tensor $\mathbf{T}$ in different bases, $\bar{\mathcal{B}}$ or $\mathcal{B}$, we first note that the matrix of the change of basis is $[\mathbf{R}]^T := [\mathbf{R}]^T_{\bar{\mathcal{B}}} = [\mathbf{R}]^T_{\mathcal{B}}$ (where square brackets denote component evaluation in the basis indicated as an index); therefore:

$$[\mathbf{w}]_{\mathcal{B}} = [\mathbf{R}]^T[\mathbf{w}]_{\bar{\mathcal{B}}}, \quad [\mathbf{T}]_{\mathcal{B}} = [\mathbf{R}]^T[\mathbf{T}]_{\bar{\mathcal{B}}}[\mathbf{R}] \qquad [1]$$

Components of attached vectors and tensors in their "natural bases"

To stress the independence of the two concepts previously discussed, both vectors $\mathbf{r}$, $\bar{\mathbf{r}}$, as well as both the tensors $\mathbf{L}$, $\bar{\mathbf{L}}$, could be represented either in $\bar{\mathcal{B}}$ or $\mathcal{B}$, and denoted by $[\bar{\mathbf{r}}]_{\bar{\mathcal{B}}}$, $[\bar{\mathbf{r}}]_{\mathcal{B}}$, $[\mathbf{r}]_{\bar{\mathcal{B}}}$, $[\mathbf{r}]_{\mathcal{B}}$,..., respectively. Of course, it appears "more natural" to express $\bar{\mathbf{r}}$, $\bar{\mathbf{L}}$ in $\bar{\mathcal{B}}$ and $\mathbf{r}$, $\mathbf{L}$ in $\mathcal{B}$ since, in those bases, they are more meaningful. When we apply the component transformations to the attached vectors, we have:

$$[\mathbf{r}]_{\mathcal{B}} = [\mathbf{R}]^T[\mathbf{r}]_{\bar{\mathcal{B}}} = [\mathbf{R}]^T[\mathbf{R}\bar{\mathbf{r}}]_{\bar{\mathcal{B}}} = [\bar{\mathbf{r}}]_{\bar{\mathcal{B}}} \qquad [2]$$

and, for tensors:

$$[\mathbf{L}]_{\mathcal{B}} = [\mathbf{R}]^T[\mathbf{L}]_{\bar{\mathcal{B}}}[\mathbf{R}] = [\mathbf{R}]^T[\mathbf{R}\bar{\mathbf{L}}\mathbf{R}^T]_{\bar{\mathcal{B}}}[\mathbf{R}] = [\bar{\mathbf{L}}]_{\bar{\mathcal{B}}} \qquad [3]$$

In conclusion, as expected, we find that *the attached vectors and tensors have the same components in their respective "natural bases"*.

Notation adopted

In this book, with a few exceptions to be clearly stated later, an overbar affixed on a vector or tensor denotes that that entity is attached to the basis $\bar{\mathcal{B}}$; the same symbol without a bar denotes that the vector or tensor has been transformed by a rotation which led $\bar{\mathcal{B}}$ to $\mathcal{B}$. When, instead, an overbar appears on column matrices, matrices, or their components, it denotes that a vector or tensor (regardless if it is attached to the basis or not) has been represented in $\bar{\mathcal{B}}$, and, without a bar, in $\mathcal{B}$. As an example, $\bar{\boldsymbol{w}} := [\mathbf{w}]_{\bar{\mathcal{B}}}$ and $\boldsymbol{w} = [\mathbf{w}]_{\mathcal{B}}$, which are related by $\boldsymbol{w} = \boldsymbol{R}^T\bar{\boldsymbol{w}}$; similarly, $\bar{\boldsymbol{T}} := [\mathbf{T}]_{\bar{\mathcal{B}}}$ and $\boldsymbol{T} = [\mathbf{T}]_{\mathcal{B}}$, related by $\boldsymbol{T} = \boldsymbol{R}^T\bar{\boldsymbol{T}}\boldsymbol{R}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

List of Main Symbols

a_i	amplitude of the i-th distorsional mode
$\mathbf{a}_i, \bar{\mathbf{a}}_i$	unit vectors of the bases $\mathcal{B}$ and $\bar{\mathcal{B}}$ $(i = 1, 2, 3)$, respectively
$\mathbf{a}_\alpha, \bar{\mathbf{a}}_\alpha$	unit vectors of the bases $\mathcal{B}_f, \bar{\mathcal{B}}_f$ $(\alpha = t, n, b)$, respectively
A	cross-section area
A_i	shear-areas of the cross-section $(i = 2, 3)$
$\boldsymbol{A}$	velocity constraint operator
$\boldsymbol{A}^\star$	equilibrium condensation operator
$\mathcal{A}$	cross-section domain
$\boldsymbol{\mathcal{A}}_H^\star$	boundary equilibrium condensation operator
b	thickness of the TWB cross-section
$\mathbf{b}$	volume force density
B_i	bi-distorsional forces
$\boldsymbol{B}_\omega, \bar{\boldsymbol{B}}_\omega$	matrices collecting the components of the spin-axes in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively
$\mathcal{B}, \bar{\mathcal{B}}, \mathcal{B}_e$	current, reference and external bases
$\mathcal{B}_f$	Frenet basis ($\bar{\mathcal{B}}_f$ also used for cables)
c	circumferential abscissa
$c_i, \bar{c}_i$	components of $\mathbf{c}$ in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively
$C_{iH}, \bar{C}_{iH}$	components of $\mathbf{C}_H$ in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively
$\mathbf{c}$	linear couple density
C	cross-section flexural- (or shear-) center
$\mathbf{C}_H$	end-couple
$\mathring{\mathbf{c}}, \mathring{\mathbf{C}}_H$	pre-existing couples
$\tilde{\mathbf{c}}, \tilde{\mathbf{C}}_H$	incremental couples
$\boldsymbol{c}, \bar{\boldsymbol{c}}$	column matrices collecting the components of $\mathbf{c}$ in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively
$\boldsymbol{C}_H, \bar{\boldsymbol{C}}_H$	column matrices of the components of $\mathbf{C}_H$ in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$\mathcal{C}$	middle-line of the TWB cross-section
$\mathbf{d}$	stretching velocity gradient
D_i	distorsional forces
$\boldsymbol{D}$	kinematic operator
$\boldsymbol{D}_0$	infinitesimal kinematic operator
$\boldsymbol{D}^\star$	equilibrium operator
$\boldsymbol{D}_0^\star$	linear equilibrium operator
$\boldsymbol{\mathcal{D}}_H^\star$	boundary equilibrium operator
$\boldsymbol{\mathcal{D}}_{0H}^\star$	linear boundary equilibrium operator
e	unit extension
$\mathbf{e}$	(reference, or right) strain vector
$\boldsymbol{e}$	column matrix collecting the components of $\mathbf{e}$ in $\bar{\mathcal{B}}$
E	Young modulus
E_{ij}	elastic coefficients
$\boldsymbol{E}$	elastic matrix
$\boldsymbol{\mathcal{E}}$	column matrix collecting the strain-displacement relationships
$\mathbf{f}$	surface force density
G	cross-section centroid, *or*
G	tangential elastic modulus
$\boldsymbol{G}$	geometric stiffness operator
$\boldsymbol{\mathcal{G}}_H$	boundary geometric stiffness operator
$\mathbf{h}$	angular momentum per unit length
H	dummy index $H = A, B$ denoting the end points of the body
$\mathcal{H}$	Hamilton functional
$\mathbf{i}_j$	unit vectors of the external basis
I_α	mass-moment of inertia of the cross-section
$\mathbf{I}_G$	inertia tensor
$\boldsymbol{I}$	identity matrix
$\mathbf{j}$	linear momentum per unit length
J_α	geometrical characteristics of the cross-section
$k, \bar{k}$	Frenet curvature, in the current and reference configuration, respectively
$\mathbf{k}, \bar{\mathbf{k}}$	axial vectors of $\mathbf{K}, \bar{\mathbf{K}}$, respectively
$\boldsymbol{k}, \bar{\boldsymbol{k}}$	column matrices collecting the components of $\mathbf{k}$ and $\bar{\boldsymbol{k}}$ in $\bar{\mathcal{B}}$, respectively
$\mathbf{K}$	(reference, or right) curvature in the current state
$\bar{\mathbf{K}}$	curvature in the reference state (or initial curvature)
$\boldsymbol{K}, \bar{\boldsymbol{K}}$	matrices collecting the components of $\mathbf{K}, \bar{\mathbf{K}}$ in $\bar{\mathcal{B}}$, respectively
l	initial length of the beam

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$\boldsymbol{L}$	linear elastic stiffness operator
$\boldsymbol{\mathcal{L}}_H$	boundary elastic stiffness operator
m	mass per unit length
$\mathbf{m}$	couple-stress
$\mathring{\mathbf{m}}$	pre-existing couple-stress
$\tilde{\mathbf{m}}$	incremental couple-stress
$\boldsymbol{m}$	column matrix collecting the components of $\mathbf{m}$ in $\mathcal{B}$
M_i	torsional $(i=1)$ and bending $(i=2,3)$ moment components in $\mathcal{B}$
$\bar{\mathbf{n}}$	rotation axis
N	axial (or normal) force component in $\mathcal{B}$
$p_i, \bar{p}_i$	components of $\mathbf{p}$ in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively
$P_{iH}, \bar{P}_{iH}$	components of $\mathbf{P}_H$ in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively
$\mathbf{p}$	linear force density
$\mathbf{P}_H$	end-force
$\mathring{\mathbf{p}}, \mathring{\mathbf{P}}_H$	preloads
$\tilde{\mathbf{p}}, \tilde{\mathbf{P}}_H$	incremental loads
$\boldsymbol{p}, \bar{\boldsymbol{p}}$	column matrices of the components of $\mathbf{p}$ in $\mathcal{B}$ and $\bar{\mathcal{B}}$
$\boldsymbol{P}_H, \bar{\boldsymbol{P}}_H$	column matrices of the components of $\mathbf{P}_H$ in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively
$\mathcal{P}_{int}, \mathcal{P}_{ext}$	virtual power, internal and external
q_i	distortional linear force density
Q_i	distortional end-force
r_i	components of $\mathbf{r}$ in $\mathcal{B}$ or, equivalently, of $\bar{\mathbf{r}}$ in $\bar{\mathcal{B}}$ $(i=2,3)$
$\mathbf{r}, \bar{\mathbf{r}}$	oriented distance of a point on the cross-section from the centroid, in the current or reference configuration, respectively
$\mathbf{R}$	rotation tensor
$\breve{\mathbf{R}}_H$	rotation prescribed at the ends
$\boldsymbol{R}$	rotation matrix, collecting the components of $\mathbf{R}$ in $\bar{\mathcal{B}}$ and $\mathcal{B}$
s	axis-abscissa
$\mathbf{s}$	spin-gradient
$\mathcal{S}$	beam- or cable-axis
t	time
$\mathbf{t}$	force-stress
$\mathring{\mathbf{t}}$	pre-existing force-stress
$\tilde{\mathbf{t}}$	incremental force-stress
$\boldsymbol{t}$	column matrix collecting the components of $\mathbf{t}$ in $\mathcal{B}$
T	tension of the cable, *or*
T	kinetic energy
T_i	shear-force components in $\mathcal{B}$ $(i=2,3)$
u_i	displacement components in $\bar{\mathcal{B}}$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$\mathbf{u}$	displacement
$\breve{\mathbf{u}}_H$	displacement prescribed at the ends
$\boldsymbol{u}$	column matrix collecting the components of $\mathbf{u}$ in $\bar{\mathcal{B}}$
U	elastic potential energy
v_i	velocity component in $\bar{\mathcal{B}}$
$\mathbf{v}$	velocity
$\boldsymbol{v}$	column matrix collecting the components of $\mathbf{v}$ in $\bar{\mathcal{B}}$
$\boldsymbol{w}$	column matrix collecting the configuration variables
W	external work
$\mathbf{W}$	spin tensor
$\boldsymbol{W}$	matrix collecting the component of $\mathbf{W}$in $\mathcal{B}$
$\boldsymbol{\mathcal{W}}$	column matrix collecting the constraint relationships of internally constrained models
$\mathbf{x}$	position vector in the current configuration
$\bar{\mathbf{x}}$	position vector in the reference configuration
$\boldsymbol{y}_\alpha$	column vector collecting Y_α
Y_α	geometric characteristics of a TWB cross-section
α_i	distortional strain
β_i	distorsion gradient
γ_i	transverse-strain components in $\bar{\mathcal{B}}$ $(i = 2, 3)$
$\boldsymbol{\gamma}$	shear-strain vector in the fiber-model
$\mathit{\Gamma}_w, \mathit{\Gamma}_{wC}$	scalar geometrical characteristics of a TWB cross-section
$\mathbf{\Gamma}$	tensor geometrical characteristic of a TWB cross-section
δ	deviation angle between the Frenet and the principal triad in a curved beam
ε	longitudinal strain component in $\bar{\mathcal{B}}$
$\boldsymbol{\varepsilon}$	column matrix collecting the strains
ζ, η	vicosity coefficients
θ_i	Tait-Bryan angles, or elementary rotations
$\boldsymbol{\theta}$	column matrix collecting the Tait-Bryan angles (pseudo-rotation vector)
$\mathit{\Theta}$	rotation angle around the rotation axis $\bar{\mathbf{n}}$
$\kappa_i, \bar{\kappa}_i$	components of $\mathbf{k}$, $\bar{\mathbf{k}}$ in $\bar{\mathcal{B}}$
λ	Lagrangian multiplier, *or*
λ	stretch
$\mathbf{\Lambda}$	skew-symmetric tensor whose axial vector is $\mathbf{x}'$
$\boldsymbol{\mathit{\Lambda}}$	scalar representation of $\mathbf{\Lambda}$ in $\mathcal{B}$
$\mathit{\Pi}$	total potential energy
ρ	volume mass density
ρ_G, ρ_C	area inertia radius, with respect to G or C

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

σ	stress component normal to the cross section in 3D-models
$\boldsymbol{\sigma}$	column matrix collecting the stresses
τ_α	stress component tangent to the cross-section in 3D-models
χ_i	components of $\boldsymbol{\chi}$ in $\bar{\mathcal{B}}$
$\boldsymbol{\chi}$	axial vector of $\mathbf{X}$
$\mathbf{X}$	change (or increment) of curvature in a curved beam
ϕ	elastic potential energy density
ψ_{ij}	component of $\boldsymbol{\psi}_i$ in $\mathcal{B}$ or, equivalently, of $\bar{\boldsymbol{\psi}}_i$ in $\bar{\mathcal{B}}$
$\boldsymbol{\psi}_i, \bar{\boldsymbol{\psi}}_i$	distortional modes of a TWB, expressed in the current or reference configuration, respectively
$\omega_i, \bar{\omega}_i$	components of $\boldsymbol{\omega}$ in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively
$\boldsymbol{\omega}$	spin-vector, axial vector of $\mathbf{W}$
$\boldsymbol{\omega}, \bar{\boldsymbol{\omega}}$	column matrices collecting the components of $\boldsymbol{\omega}$ in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively
Ω_C	sectorial area with respect to C
$\boldsymbol{\Omega}$	infinitesimal rotation matrix

Indeces

a, r	active/reactive
c, u	constrained/unconstrained
c, r	current/reference
m, s	master/slave
w, π	out-of-plane/in-plane components

Overmarks

$(\cdot)'$	space-derivative
$\dot{(\cdot)}$	time-derivative
$\mathring{(\cdot)}$	preload and prestress
$\tilde{(\cdot)}$	incremental load or stress
$\breve{(\cdot)}_H$	prescribed displacement/rotation at the boundary
$\bar{(\cdot)}$	vector or tensor attached to $\bar{\mathcal{B}}$, *or*
$\bar{(\cdot)}$	column matrix or matrix representing the components in $\bar{\mathcal{B}}$ of a vector or tensor
$\hat{(\cdot)}$	deformed geometric entity

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Chapter 1

A One-Dimensional Beam Metamodel

We introduce a one-dimensional (1D) *metamodel* of a beam as a progenitor of specific models to be formulated later in the book. The metamodel establishes properties and rules that highlight the common logic structure of the particular models. It leads us to formulate equations in terms of abstract quantities (typically column-vectors and formal matrix differential operators), whose contents do not need to be specified at this stage. We first address internally unconstrained beams, i.e. models in which all the variables introduced in the kinematic description are not subject to additional limitations. Formulation of the balance equations via the virtual power principle (VPP) straightforwardly leads us to recognize kinematic and equilibrium operators as mutually adjoint. Then, we analyze internally constrained beams, in which one or more strains are prescribed to identically vanish along the beam, for which we illustrate two alternative approaches: (a) the *mixed formulation*, in which reactive stresses enter the set of the main unknowns; and (b) the *displacement formulation*, in which kinematic and dynamic equations are condensed in order to satisfy constraints and to filter reactive stresses, respectively. Then, prestressed beams are considered, for which the reference state differs from the natural state, since stresses there are different from zero for the existence of preloads. Both cases of internally unconstrained and constrained prestressed beams are analyzed, and the previous analysis is entirely repeated to account for prestress. In this context, attention is devoted to the linearized theory, widely used in technical applications, able to furnish critical loads (in buckling problems), eigenfrequencies (of strings and cables), as well as the response to small incremental loads. For each

Mathematical Models of Beams and Cables Angelo Luongo and Daniele Zulli
© 2013 ISTE Ltd. Published 2013 by ISTE Ltd.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

problem addressed, a brief sketch of the variational formulation is also given as an alternative approach.

1.1 Models and metamodel

In the modeling process of the mechanical behavior of a beam or cable, different phenomenological aspects can be taken into account, and/or the same aspect described at different sophistication levels. Thus, a beam can be 3D or 1D, with rigid or deformable cross-sections, with deformability permitted in the plane and/or out-of-plane of the section. Each of these assumptions leads to a specific *model*; thus, for example, we have the "Timoshenko beam", which is able to describe the relative rotation between the rigid cross-section and the centerline (the so-called *shearable beam*), as well as the "Euler–Bernoulli beam", in which the cross-section keeps its orthogonality to the centerline (the so-called *unshearable beam*), or the "Vlasov thin-walled beam", in which the cross-section is allowed to warp, but not to deform in its plane, or the "Brazier tubular beam" which does not warp, but ovalizes itself. As a further example, a cable can be considered as flexible, and therefore modeled as a (prestressed) Cauchy continuum (the "flexible cable"), or provided with flexural and torsional rigidity, and therefore modeled as a Cosserat continuum (the "stiff cable").

All these mathematical models, although different, and therefore leading to different equations, have common features, which refer to the logic underlying all of them. It is therefore convenient to introduce a *metamodel* (from the Greek "beyond the model"), which is independent of the specific aspects of the single model, but, in contrast, highlights the common structure of the models. A quite accepted definition of a metamodel is the following: "a precise definition of the constructs and rules needed for creating specific models". Accordingly, the metamodel is a system of inter-related "empty boxes"; once it is available, formulation of specific models consists of "filling in" these boxes.

As in all the problems of continuum mechanics, modeling requires analyzing three independent aspects: (a) geometrical (or kinematic), (b) dynamical, and (c) constitutive. Here, we will introduce these three aspects from an abstract point of view, in order to formulate a metamodel. However, to make the discussion clearer, we will often refer to (linear) models known to the reader, with the only purpose of exemplification. Although the metamodel would work for any spatial dimension, we will refer to a 1D problem because this will be the object of our successive studies.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

1.2 Internally unconstrained beams

Let us consider a 1D deformable body, whose material points P densely fill a curve in the space. We will say that the beam is *locally rigid*, when P is capable of translations (non-polar continuum) and, possibly, also of rotations (polar continuum), i.e. it behaves as an evanescent rigid body. We will say that the beam is *locally non-rigid* when P is also endowed with a "shape" susceptible to change in time. Standard models of cables and beams, possessing rigid cross-sections, fall into the first category; non-standard models, accounting for the change of shape of the cross-section, fall into the second category.

We will use the wording "position of the point P" in a generalized sense, including place, attitude and "shape" of the point. The collection of the positions is called a *configuration*. The configuration assumed by the body at $t = 0$ is called the *reference configuration*; the one assumed at time t is called the *current configuration*. Let us consider a curve $\mathcal{S}$, of extremes A, B, on which the body lies at $t = 0$, and let $s \in [0, l]$ be a curvilinear abscissa taken on it; in such a way s is a label for the material point P, in the sense that $\mathcal{Q}(s, t)$ represents the value assumed by the quantity $\mathcal{Q}$ at the position occupied by the material point P at time t.

1.2.1 Kinematics

To describe the current configuration of the body, we follow the referential description of the continuum mechanics, by introducing suitable *generalized displacements* $\boldsymbol{w}(s,t) := (w_i\,(s,t))^T$, $i = 1, \ldots, N$, measured from the reference configuration. This is a set of kinematic descriptors (translations, rotations and distortion parameters) able to describe the change of position of the point P in passing from the reference to the current configuration. The integer N is also called the *number of degrees of freedom* of the point.

The change of configuration, except for special rigid transformations, entails a change of shape of the body, which we will call a *deformation*[1]. A measure of the *local* change of shape is called a *strain*; examples of strains are not only extension, shear, flexure and torsion of a bar, but also warping and ovalization of a pipe cross-section. These can be collected in the column-matrix of *generalized strains* $\boldsymbol{\varepsilon}(s,t) := (\varepsilon_j\,(s,t))^T$, with $j = 1, \ldots, M$. The number of generalized strains is closely related to the number of generalized displacements and the dimensions of the space in which the body is embedded. As a matter of fact, the change of shape of an infinitesimal element of length ds depends on the displacements at its end, $\boldsymbol{w}(s,t)$ and $\boldsymbol{w}(s+ds,t)$;

1. We prefer to reserve the word deformation to *non-rigid* transformations, although it is used in the literature with a wider meaning.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

the latter, in turn, can be expressed by Taylor series as $\boldsymbol{w}(s,t) + \boldsymbol{w}'(s,t)ds$[2], so that the configuration depends on $2N$ independent quantities, namely $(\boldsymbol{w}(s,t),\ \boldsymbol{w}'(s,t))$[3]. Since R of them describe a rigid motion, the number of strains is $M := 2N - R$. Usually, $R = 6, 3, 1$ in the spatial, planar and linear cases, respectively; however, it can be lower, if some rotation remains undefined when $\boldsymbol{w}(s,t)$ and $\boldsymbol{w}'(s,t)$ are prescribed. For example, for a Timoshenko beam in the space, it is $N = 6$ (three translations and three rotation angles) and $R = 6$, so that $M = 6$ (one extension, two shear strains and three curvatures); in the plane, it is $N = 3$ (two translations and one rotation) and $R = 3$, and therefore $M = 3$ (one extension, one shear strain and one curvature). For a flexible cable in the plane, it is $N = 2$ (two translations) and $R = 3$, so that $M = 1$ (the extension); however, in the spatial case, it is $N = 3$ (three translations) but $R = 5$, since the displacements and their derivatives are unable to describe the rotation around the tangent to the element, so that it is still $M = 1$ (the extension).

The generalized strains depend on displacements and their first derivatives via the *strain–displacement relationships*. These are *nonlinear* differential equations of the type:

$$\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right) \qquad [1.1]$$

where the arguments s, t have been understood. Displacements and strains related by equation [1.1] are called *kinematically admissible*.

The kinematic description is completed by the geometric boundary conditions, which prescribe (part of) the displacements at the ends, where mechanical devices are applied, namely:

$$\boldsymbol{w}_H\left(t\right) = \breve{\boldsymbol{w}}_H\left(t\right), \qquad H = A, B \qquad [1.2]$$

where the overmark denotes a known term.

However, not only strains, but even *strain-rates* are of interest. They are obtained by time-differentiating equation [1.1], thus obtaining[4]:

$$\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}\left(\boldsymbol{w}, \boldsymbol{w}'\right)\dot{\boldsymbol{w}} \qquad [1.3]$$

where[5]:

$$\boldsymbol{D} := \frac{\partial \boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right)}{\partial \boldsymbol{w}} + \frac{\partial \boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right)}{\partial \boldsymbol{w}'}\frac{\partial}{\partial s} \qquad [1.4]$$

2. Here and further on a dash denotes s-differentiation.

3. There exist richer continua of *higher gradient* (see e.g. [DEL 09]), which call for higher derivatives of $\boldsymbol{w}(s,t)$, which, however, we will not consider in this book.

4. Here and further on a dot denotes t-differentiation.

5. The derivative of a vector with respect to a vector is a matrix; thus, e.g. $\frac{\partial \boldsymbol{\mathcal{E}}}{\partial \boldsymbol{w}} = \left[\frac{\partial \mathcal{E}_i}{\partial w_j}\right]$, where i is the row index and j the column index.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

is a (formal) $M \times N$ matrix, containing the displacements and the space-differential operator $\partial_s := \partial/\partial s$[6]. $\boldsymbol{D}$ is a linear differential operator which transforms the generalized velocities in generalized strain-rates, called the *kinematic operator*. This depends, via $\boldsymbol{w}$ and $\boldsymbol{w}'$, on the configuration assumed by the body at time t.

If we consider an infinitesimal time interval dt, and we want to evaluate the strains that have been experienced by the beam in this interval, we have:

$$\delta\boldsymbol{\varepsilon} = \boldsymbol{D}\left(\boldsymbol{w}, \boldsymbol{w}'\right)\delta\boldsymbol{w} \qquad [1.5]$$

where $\delta\boldsymbol{\varepsilon} := \dot{\boldsymbol{\varepsilon}}dt$ are infinitesimal strains produced by infinitesimal displacements $\delta\boldsymbol{w} := \dot{\boldsymbol{w}}dt$, superimposed to a deformed state $\boldsymbol{w}$. If we take $\boldsymbol{w} \equiv \boldsymbol{0}$, i.e. if we consider infinitesimal displacements undergone by the beam starting from its undeformed state, then the *infinitesimal kinematic operator* $\boldsymbol{D}_0 := \boldsymbol{D}\left(\boldsymbol{0}, \boldsymbol{0}\right)$ must be considered, which is well-known in the linear theory.

Boundary conditions for equation [1.3] are $\dot{\boldsymbol{w}}_H = \boldsymbol{0}$ and for equation [1.5] are $\delta\boldsymbol{w}_H = \boldsymbol{0}$.

1.2.2 Dynamics

Dynamics concerns the study of the contact internal actions exchanged by adjacent points, when the body is loaded by external (active, dissipative or inertial) forces. The internal action is described by *generalized stresses* (forces and couples in locally rigid beams, but also more complex actions like the "bimoment", in locally non-rigid beams). The relationships linking generalized stresses and external forces are called *balance equations* (or equilibrium equations, when they refer to the static case). To define generalized stresses and to derive balance equations, we can follow two alternative philosophies, both popular in the literature: (a) the *power balance formulation*, based on the "virtual power principle", or (b) the *force balance formulation*, based on the "momentum principles" (or cardinal equations of motion, or, in the static case, equilibrium equations)[7]. Both the approaches are (not independent) postulates of the continuum mechanics, leading to the same results, so that we can choose which of them to use. However, if the choice is just a question of taste when dealing with locally rigid beams, the first approach is mandatory when locally non-rigid beams are addressed in the context of 1D models because the cardinal equations are not sufficient to describe the motion of a non-rigid body[8]. In

6. As an example, for a rod embedded in a 1D space, we have $\varepsilon = w_1'$, hence $\boldsymbol{D} = (\partial_s)$.

7. These two approaches are also known in literature as *integral (or weak) formulation*, and *differential (or strong) formulation*, respectively.

8. Of course, the force balance approach could be used for a 3D model, as for example is usually done in the de Saint-Venant Problem.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

this book, we use both the approaches, as discussed later, guided by convenience reasons.

The virtual power principle

We consider a beam loaded by generalized *external* forces (possibly including inertia and damping forces), acting in the domain, of linear density $\boldsymbol{p} := (p_i(s,t))^T$ (with $i = 1, \ldots, N$, i.e. a force component for each degree of freedom, d.o.f), as well as boundary external forces $\boldsymbol{P}_H := (P_{iH}(s,t))^T$, applied at $H = A, B$. These forces, except the trivial (but frequent) case of dead loads, depend on the configuration (e.g. if they are of follower type), for which $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{w}),\ \boldsymbol{P}_H = \boldsymbol{P}_H(\boldsymbol{w})$, although we will often understand the argument. We then consider the beam *frozen* in the current configuration and superimpose on it a *virtual motion* (i.e. a motion unrelated to the forces), made of a velocity field $\dot{\boldsymbol{w}}$ and a strain-rate field $\dot{\boldsymbol{\varepsilon}}$. The following quantities are introduced:

$$
\begin{aligned}
\mathcal{P}_{ext} &:= \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \\
\mathcal{P}_{int} &:= \int_{\mathcal{S}} \boldsymbol{\sigma}^T \dot{\boldsymbol{\varepsilon}} ds
\end{aligned}
\qquad [1.6]
$$

called the “external virtual power” and the “internal virtual power” of the beam, respectively. The first of them is the usual definition of the power of a system of forces, except for the fact that forces and velocities are unrelated. In the second definition, $\boldsymbol{\sigma} := (\sigma_j(s,t))^T$, with $j = 1, \ldots, M$, are *generalized stresses*. According to this approach, no physical meaning is given to them, but, in analogy with external forces and velocities, they must be recognized as the dynamic action *dual* of the strain-rate (which, in contrast, do have a geometrical meaning).

The VPP establishes that, *in any kinematically admissible virtual motion* $(\dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}})$, *the external virtual power spent by the generalized forces* $\boldsymbol{p}, \boldsymbol{P}_H$ *on the velocity field* $\dot{\boldsymbol{w}}$*, equates the internal virtual power spent by the stresses* $\boldsymbol{\sigma}$ *on the strain-rate field* $\dot{\boldsymbol{\varepsilon}}$, i.e.:

$$
\int_{\mathcal{S}} \boldsymbol{\sigma}^T \dot{\boldsymbol{\varepsilon}} ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \qquad \forall (\dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}) \,|\, \dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}\dot{\boldsymbol{w}} \qquad [1.7]
$$

The VPP furnishes the balance equations via the following procedure. By using equation [1.3], and integrating by parts to “free” the velocities by the derivatives, the

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

internal power reads:

$$\int_{\mathcal{S}} \boldsymbol{\sigma}^T \boldsymbol{D}\dot{\boldsymbol{w}} ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{D}^{\star} \boldsymbol{\sigma} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{\mathcal{D}}_H^{\star} \boldsymbol{\sigma} \qquad [1.8]$$

where the following operators have been introduced, accounting for equation [1.4][9]

$$\begin{aligned} \boldsymbol{D}^{\star} &:= \left(\frac{\partial \boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}')}{\partial \boldsymbol{w}} - \frac{\partial}{\partial s} \left(\frac{\partial \boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}')}{\partial \boldsymbol{w}'} \right) - \frac{\partial \boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}')}{\partial \boldsymbol{w}'} \frac{\partial}{\partial s} \right)^T \\ \boldsymbol{\mathcal{D}}_H^{\star} &:= \mp \left(\frac{\partial \boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}')}{\partial \boldsymbol{w}'} \right)_H^T \end{aligned} \qquad [1.9]$$

The VPP, therefore, reads:

$$\int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \left(\boldsymbol{D}^{\star} \boldsymbol{\sigma} - \boldsymbol{p} \right) ds - \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \left(\boldsymbol{\mathcal{D}}_H^{\star} \boldsymbol{\sigma} - \boldsymbol{P}_H \right) = 0 \qquad \forall \dot{\boldsymbol{w}} \qquad [1.10]$$

and, since $\dot{\boldsymbol{w}}$ is arbitrary, it leads to the following field equations:

$$\boldsymbol{D}^{\star}(\boldsymbol{w}, \boldsymbol{w}') \, \boldsymbol{\sigma} = \boldsymbol{p} \qquad [1.11]$$

and to the boundary conditions:

$$\left[\dot{\boldsymbol{w}}^T \left(\boldsymbol{\mathcal{D}}^{\star}(\boldsymbol{w}, \boldsymbol{w}') \, \boldsymbol{\sigma} - \boldsymbol{P} \right) \right]_H = \boldsymbol{0} \qquad H = A, B \qquad [1.12]$$

Equation [1.11] is the balance (or equilibrium) equation sought. Equation [1.12] supplies the relevant boundary conditions, called mechanical (or natural) conditions. They supplement the geometric (or essential) boundary conditions [1.2] in the following senses: (a) if H is fully constrained, then $\dot{\boldsymbol{w}}_H = \boldsymbol{0}$, and therefore no mechanical conditions hold there; (b) if H is fully free, then $\dot{\boldsymbol{w}}_H \neq \boldsymbol{0}$ and it is arbitrary, and therefore $\boldsymbol{\mathcal{D}}_H^{\star} \boldsymbol{\sigma} = \boldsymbol{P}_H$ must hold there. Similar properties hold for partially restrained ends. In conclusion, if a displacement component is prescribed, no mechanical condition must be added; if a displacement component is free, a scalar mechanical condition must be enforced. Therefore, geometric and mechanical conditions are alternative.

The operator $\boldsymbol{D}^{\star}(\boldsymbol{w}, \boldsymbol{w}')$, which appears in the balance equations, is a formal $N \times M$ matrix, depending on the operator ∂_s[10]. It is a linear differential operator that

9. For example, $\int_{\mathcal{S}} \boldsymbol{\sigma}^T \left(\boldsymbol{A} \frac{d}{ds} \right) \dot{\boldsymbol{w}} ds = -\int_{\mathcal{S}} \frac{d}{ds} \left(\boldsymbol{\sigma}^T \boldsymbol{A} \right) \dot{\boldsymbol{w}} ds + \left[\boldsymbol{\sigma}^T \boldsymbol{A} \dot{\boldsymbol{w}} \right]_A^B = -\int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \frac{d}{ds} \left(\boldsymbol{A}^T \boldsymbol{\sigma} \right) ds + \left[\dot{\boldsymbol{w}}^T \boldsymbol{A}^T \boldsymbol{\sigma} \right]_A^B$.

10. As an example, for a rod embedded in a 1D space, the equilibrium equation reads $N' + p_1 = 0$, hence $\boldsymbol{D}^{\star}(\cdot) = (-\partial_s)$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

transforms the generalized stresses into generalized forces, and it is called the *equilibrium operator*. Note that it depends on the state $\boldsymbol{w}$ because it describes the equilibrium of the beam in the current configuration, thus encompassing the nonlinear nature of the problem. When, in contrast, the effects of deformation are ignored (i.e. the current configuration is confused with the reference configuration), then the equilibrium operator reduces to $\boldsymbol{D}_0^\star := \boldsymbol{D}^\star(\mathbf{0},\mathbf{0})$, which is the well-known *linear equilibrium operator* of the linear theory. The operators $\boldsymbol{\mathcal{D}}_H^\star$ are (algebraic) *boundary equilibrium operators*.

REMARK 1.1. The VPP expression [1.8] is also called, in a wider context, the *extended Green identity*. It states that *the kinematic operator and the equilibrium operator, as well as the associated boundary conditions, are mutually adjoint*. Such an occurrence is called *duality property*. It is well-known in the linear field (where it concerns the adjointness property between $\boldsymbol{D}_0$ and $\boldsymbol{D}_0^\star$), but it also holds in the nonlinear field, when use is made of the kinematic and equilibrium operators relevant to the current configuration.

REMARK 1.2. The VPP could be reformulated as a *virtual work* principle (VWP). Indeed, it is sufficient to multiply both members of equation [1.7] by an infinitesimal time interval dt, and to refer to infinitesimal displacements $\delta\boldsymbol{w} := \dot{\boldsymbol{w}}dt$ and infinitesimal strains $\delta\boldsymbol{\varepsilon} := \dot{\boldsymbol{\varepsilon}}dt$, namely:

$$\int_{\mathcal{S}} \boldsymbol{\sigma}^T\delta\boldsymbol{\varepsilon}ds = \int_{\mathcal{S}} \delta\boldsymbol{w}^T\boldsymbol{p}ds + \sum_{H=A}^{B} \delta\boldsymbol{w}_H^T\boldsymbol{P}_H \qquad \forall\left(\delta\boldsymbol{w},\delta\boldsymbol{\varepsilon}\right)|\delta\boldsymbol{\varepsilon} = \boldsymbol{D}\delta\boldsymbol{w} \quad [1.13]$$

The formulation in terms of velocities is preferred in formal treatments because it does not call for resorting to the concept of "infinitesimal" displacements and strains, which understands a series expansion.

The force balance formulation for locally rigid beams

When the local structure of the 1D beam is rigid, the force balance formulation is viable. With respect to the power balance formulation, it has the advantage to endow the stresses of a physical meaning.

We consider the internal action that two parts of the beam mutually exchange at the abscissa s and time t, and denote by $\boldsymbol{f} = (f_i(s,t))^T$, $i = 1,\ldots,N$ the generalized forces (i.e. forces and couples) acting on one of the two parts, conventionally assumed as positive. Note that the internal force components are in the same number of the degrees of freedom (translations and rotations) of the "rigid" point P. We then consider an infinitesimal element of length ds, loaded by external forces per unit length $\boldsymbol{p} := (p_i(s,t))^T$, at whose ends, internal forces $\boldsymbol{f}(s,t)$ and $\boldsymbol{f}(s+ds,t) = \boldsymbol{f}(s,t) + \boldsymbol{f}'(s,t)ds$ act. Therefore, the contact action that the element

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

exchanges with the adjacent ones depends on $2N$ independent scalar quantities, namely $\left(\boldsymbol{f}(s,t),\ \boldsymbol{f}'(s,t)\right)$.

We define *stresses* the independent internal forces able to describe the more general self-equilibrated state of the element (i.e. when $\boldsymbol{p} = \mathbf{0}$). We *postulate* that equilibrium of this elementary body is governed by the *same cardinal equations* of rigid-body mechanics. Since the scalar equilibrium equations are in number of R, i.e. one for each independent rigid motion of the element, we conclude that the self-equilibrated states are $M := 2N - R$, described by M independent stresses, i.e. stresses are in the same number of strains. We collect all the stresses in a column-matrix $\boldsymbol{\sigma} := \left(\sigma_j\left(s,t\right)\right)^T$, with $j = 1, \ldots, M$. For example, for a spatial cable, we have $2N = 6$ internal end-forces, which have to satisfy $R = 5$ independent equilibrium equations (since the sixth one, relevant to the moment with respect to the tangent axis, is trivially satisfied); hence, $M = 1$ stresses exist (namely the axial force).

When the external forces are non-zero, then the cardinal equations must express the balance of external forces and stresses acting on the element. However, just N of them are significant, the remaining $R - N$ being satisfied by the stresses alone (e.g. in the spatial cable, all the moment equations are identically satisfied, so that only the translational equilibrium has to be satisfied). In conclusion, the M generalized stresses must satisfy N (differential) field balance equations; since these are linear, they assume the form [1.11].

When the element is taken at the boundary of the beam, and this is free of constraints, then the stresses are there prescribed; a suitable linear combination of the stresses must equate the external forces at the end, as stated in equation [1.12].

1.2.3 The hyperelastic law

To complete the model, we need to introduce a *constitutive law*, able to link generalized stresses $\boldsymbol{\sigma}$ to generalized strains $\boldsymbol{\varepsilon}$, thus realizing the "bridge" between kinematics and dynamics. This topic is quite difficult, when tackled in a general context. Indeed, a general law linking stresses and strains should account for the *deformation history* of the material, this requiring a quite complex mathematical apparatus (i.e. the use either of functionals or of incremental forms in terms of strain-rates and stress-rates). This, however, is a peculiarity of plasticity; if we are, instead, interested in comparatively small strains, then we can ignore the past events, and refer just to the current state, by writing $\boldsymbol{\sigma}(s,t) = \mathcal{F}(\boldsymbol{\varepsilon}(s,t),\, t)$. The explicit dependence on time is a peculiarity of viscosity, whose brief treatment will be postponed. If we admit that the constitutive law does not depend explicitly on time, we simply write $\boldsymbol{\sigma}(s,t) = \mathcal{F}(\boldsymbol{\varepsilon}(s,t))$ or, by omitting the arguments, $\boldsymbol{\sigma} = \mathcal{F}(\boldsymbol{\varepsilon})$. This law characterizes elasticity.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

If a beam is elastic, then stresses at the abscissa s and time t only depend on strains existing at the same place at the same instant. However, this concept of elasticity (said to be of Cauchy) does not match the idealization of the perception everybody has in real life, i.e. an elastic body requires some energy to be deformed, but it entirely returns when the deformation is removed, *regardless of the way the unloading is performed*. In other words, this more refined idea of elasticity (said to be of Green, or *hyperelasticity*) expresses the *conservation of energy*, i.e. the absence of dissipation in any cyclic process the body can undergo (or, equivalently, the independence of the energy spent on any paths followed to connect two states). Since just Green-elastic bodies are of interest, very often the adjective "elastic" is used as "hyperelastic", and we will comply with this tradition.

The elastic potential

The work spent by the external forces to deform the beam in an interval of time dt is equal to $\mathcal{P}_{ext}dt$, with $\mathcal{P}_{ext}$ the *deformation* external power, formally still given by equation [1.6a], but with velocities now denoting quantities related to the real (not virtual) process. Since $\mathcal{P}_{ext} = \mathcal{P}_{int}$ for the VPP [1.7], then we can define a *deformation work for unit length* as $\frac{d}{ds}\left(\mathcal{P}_{int}dt\right) = \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}dt = \boldsymbol{\sigma}^T\delta\boldsymbol{\varepsilon}$. To evaluate the work needed to lead a unitary element of beam from the state $\boldsymbol{\varepsilon}_1$ to the state $\boldsymbol{\varepsilon}_2$, we have to integrate the linear differential form $\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon}\right)^T\delta\boldsymbol{\varepsilon}$ along a line connecting the two states in the space of the strains. The result, in general, depends on the path chosen for integration (i.e. on the sequence of the deformation imposed), unless $\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon}\right)^T\delta\boldsymbol{\varepsilon}$ is an *exact differential*, i.e. it is the differential of a scalar function $\phi(\boldsymbol{\varepsilon})$. By requiring $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})^T\delta\boldsymbol{\varepsilon} = \delta\phi(\boldsymbol{\varepsilon}) \equiv \left(\delta\phi/\delta\boldsymbol{\varepsilon}\right)^T\delta\boldsymbol{\varepsilon}$, it follows that[11]:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \frac{\delta\phi(\boldsymbol{\varepsilon})}{\delta\boldsymbol{\varepsilon}} \qquad [1.14]$$

Equation [1.14] is the hyperelastic law sought. The law *postulates* the existence of a function $\phi(\boldsymbol{\varepsilon})$, called the *density of elastic potential energy* or, simply, the *elastic potential*.

The linear law

To write the elastic law, we have to assume a suitable form for the elastic potential, and then to differentiate it. If, for example, we adopt a polynomial of degree n, we obtain a stress–strain polynomial law of degree $n-1$. Of course, the simplest choice

11. This symbolism means that the ith component of the column-matrix $\boldsymbol{\sigma}$ is equal to the derivative of the scalar ϕ with respect to the ith component of the column-matrix $\boldsymbol{\varepsilon}$, i.e. $\sigma_i = \delta\phi/\delta\varepsilon_i$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

is to take a quadratic potential, from which a linear law follows. We start by assuming a *homogeneous* quadratic polynomial, as:

$$\phi(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}^T \boldsymbol{E}\boldsymbol{\varepsilon} \qquad [1.15]$$

where $\boldsymbol{E}$ is a square matrix of constants, called the *elastic matrix*. It possesses two properties: (a) $\boldsymbol{E} = \boldsymbol{E}^T$ is symmetric because it is the matrix of a quadratic form (and therefore its non-symmetric part is unessential)[12]; and (b) $\boldsymbol{E}$ is *positive definite*, this assuring that a positive work must be spent on the body in order to deform it, i.e.: $\phi(\boldsymbol{\varepsilon}) > 0 \; \forall \boldsymbol{\varepsilon} \neq \boldsymbol{0}$. By applying equation [1.14], the *Hooke law* follows:

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon} \qquad [1.16]$$

It establishes direct proportionality between stresses and strains; moreover, it states that stresses vanish when strains vanish. Since we decided to measure the strains starting from the reference configuration, the homogeneous form of the elastic law applies when the reference configuration is stress-free, also known as *unprestressed*. Such a state is called the *natural state* of the body, whose existence is postulated. We will return on this topic in the next section, when we will account for prestresses.

1.2.4 The Fundamental Problem

The *Fundamental Problem of Elasticity* (or elastic problem), relevant to a 1D beam, is stated as follows. A beam is given under assigned loads $\boldsymbol{p}(s,t)$ acting in the domain, and displacements $\breve{\boldsymbol{w}}_H(t)$ *or* forces $\boldsymbol{P}_H(t)$ prescribed/applied at the ends $H = A, B$. We want to determine the generalized displacements $\boldsymbol{w}(s,t)$, the strains $\boldsymbol{\varepsilon}(s,t)$ and the stresses $\boldsymbol{\sigma}(s,t)$. The problem is governed by the field equations [1.1], [1.11] and [1.16] and boundary conditions [1.2] and [1.12]. Overall, there are $N + 2M$ unknowns, appearing in as many field equations.

The equations can be combined according to the *displacement method*, which consists of expressing the balance equations and the boundary conditions in terms of displacements only, by using, in the order, the elastic law and the strain–displacement relationships. The stress–displacement relationships, therefore, read:

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}') \qquad [1.17]$$

and, consequently, the balance equations and the mechanical boundary conditions transform into:

$$\begin{aligned} &\boldsymbol{D}^\star(\boldsymbol{w}, \boldsymbol{w}')\,\boldsymbol{E}\boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}') = \boldsymbol{p} \\ &\boldsymbol{\mathcal{D}}_H^\star(\boldsymbol{w}, \boldsymbol{w}')\,\boldsymbol{E}\boldsymbol{\mathcal{E}}_H(\boldsymbol{w}, \boldsymbol{w}') = \boldsymbol{P}_H \end{aligned} \qquad [1.18]$$

12. We can also say that $E_{ij} = \partial^2\phi/\partial\varepsilon_i\partial\varepsilon_j = \partial^2\phi/\partial\varepsilon_j\partial\varepsilon_i = E_{ji}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

to be joined to the geometric boundary conditions [1.2]. These equations constitute a nonlinear boundary value problem for the principal unknowns $\boldsymbol{w}$. Since the equations are nonlinear, the uniqueness of the solution is not ensured.

The linear theory

If all terms in equations [1.18] are expanded around the reference configuration, and only the leading term is taken in each expansion, we have[13]:

$$\begin{aligned}
&\boldsymbol{D}^{\star}\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{D}_0^{\star}+\text{h.o.t.}\\
&\boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{D}_0\boldsymbol{w}+\text{h.o.t.}\\
&\boldsymbol{p}\left(\boldsymbol{w}\right)=\boldsymbol{p}_0+\text{h.o.t.}\\
&\boldsymbol{\mathcal{D}}_H^{\star}\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{\mathcal{D}}_{0H}^{\star}+\text{h.o.t.}\\
&\boldsymbol{P}_H\left(\boldsymbol{w}\right)=\boldsymbol{P}_{0H}+\text{h.o.t.}
\end{aligned} \qquad [1.19]$$

where use has been made of equation [1.4], and where the index 0 denotes evaluation at $\boldsymbol{w}=\boldsymbol{0}$. As a result, equations [1.18] become:

$$\begin{aligned}
&\boldsymbol{L}\boldsymbol{w}=\boldsymbol{p}_0\\
&\boldsymbol{\mathcal{L}}_H\boldsymbol{w}=\boldsymbol{P}_{0H}
\end{aligned} \qquad [1.20]$$

where:

$$\boldsymbol{L}:=\boldsymbol{D}_0^{\star}\boldsymbol{E}\boldsymbol{D}_0, \qquad \boldsymbol{\mathcal{L}}_H:\boldsymbol{\mathcal{D}}_{0H}^{\star}\boldsymbol{E}\boldsymbol{D}_{0H} \qquad [1.21]$$

are the familiar *(tangent) stiffness operators* (in the domain and at the boundary) of the linear theory. Note that $\boldsymbol{L}$ is self-adjoint, for the duality property and the symmetry of $\boldsymbol{E}=\boldsymbol{E}^T$[14].

REMARK 1.3. A consistent first-order expansion would also require the accounting of the first derivative of the loads, but this effect is ignored in the linear theory because equilibrium is referred to the reference configuration. Therefore, any information about dependence of the loads on displacements is lost.

1.3 Internally constrained beams

It is well-known, from Lagrangian mechanics, that internal constraints reduce the number of d.o.f. of the system. Thus, a collection of N particles, free in the space,

13. Here, and further on, h.o.t. denotes "higher order terms".

14. Namely, $\boldsymbol{L}^{\star}=(\boldsymbol{D}_0^{\star}\boldsymbol{E}\boldsymbol{D}_0)^{\star}=\boldsymbol{D}_0^{\star}\boldsymbol{E}^T\boldsymbol{D}_0=\boldsymbol{L}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

possesses $3N$ d.o.f., but these reduce to 6 if the mutual distances among the particles are constrained to remain unaltered in any motions, i.e. if the system behaves as a rigid body. In addition, it is also well-known that introducing internal constraints, while simplifying kinematics, makes the study of dynamics more difficult because part of the forces are unrelated to displacements. Accordingly, we say that the internal forces are *active*, when they depend on kinematic quantities via a constitutive law, and *reactive*, when they are independent of them. Thus, by using again the example of a collection of particles and assuming that they attract each other, the internal forces depend on the mutual distances if the particles are unconstrained, but they assume any magnitude if the particles are rigidly connected.

The degenerateness of the constitutive law can also be understood if we consider a linear spring, whose stiffness quasi-statically increases to infinite. Until the stiffness is finite, there exists proportionality between the force and the elongation (i.e. the response curve is a straight line, whose slope is the stiffness); however, when the stiffness becomes infinitely large, any force can be obtained because the product of infinite (the stiffness) by zero (the elongation) is undetermined (the response curve is vertical, i.e. it is the graph of a degenerate, not single-valued, function). As a matter of fact, in the limit process, the spring becomes a rigid truss, able to supply any force aligned with its axis.

These ideas can be translated into the mechanics of a deformable body, in particular beams. We are encouraged to formulate internally constrained models, in which the configuration variables are not free, but are required to satisfy one or more geometrical constraints. Although, in principle, any conditions could be introduced, the most meaningful of them consist of *vanishing one or more strains*, identically along the beam. Thus, a beam is inextensible (or unshearable) if the elongation (or the shear-strain) is prevented. Later in the book (Chapter 4), we will discuss the conditions under which constrained models are applicable to real cases, and also consider more complex linear and nonlinear constraint conditions (Chapter 8).

Because of the internal constraints, the dual stresses (i.e. those spending power on the constrained strains) become reactive, so that they cannot be expressed by an elastic law. This drawback calls either for a mixed displacement–stress formulation, or for a special treatment to eliminate reactive stresses, which we are going to illustrate in the next sections.

1.3.1 The mixed formulation for the internally constrained beam kinematics and constraints

The deformation of a beam is described by M independent quantities $\boldsymbol{\varepsilon}(s,t)$. We assume that $M_c < M$ of them are identically zero, i.e. $\boldsymbol{\varepsilon}_c(s,t) = \mathbf{0}\ \forall (s,t)$, and we

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

call them the *constrained strains*. The remaining $M_u := M - M_c$ strains, $\boldsymbol{\varepsilon}_u \neq \mathbf{0}$, are referred to as the *unconstrained* (or admissible) *strains*. As a result, $\boldsymbol{\varepsilon} := (\boldsymbol{\varepsilon}_u, \boldsymbol{\varepsilon}_c)^T$, and equations [1.1] read:

$$\begin{pmatrix} \boldsymbol{\varepsilon}_u \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{E}}_u (\boldsymbol{w}, \boldsymbol{w}') \\ \boldsymbol{\mathcal{E}}_c (\boldsymbol{w}, \boldsymbol{w}') \end{pmatrix} \qquad [1.22]$$

with the boundary conditions $\boldsymbol{w}_H = \breve{\boldsymbol{w}}_H$, where $H = A, B$. We will refer to the upper part of equations [1.22] as the strain–displacement relationships, and to the lower part as *a set of constraints*, limiting the arbitrariness of the displacements and their derivatives.

By time-differentiating equations [1.22], we obtain the strain-rate-velocity relationships:

$$\begin{pmatrix} \dot{\boldsymbol{\varepsilon}}_u \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{D}_u (\boldsymbol{w}, \boldsymbol{w}') \\ \boldsymbol{D}_c (\boldsymbol{w}, \boldsymbol{w}') \end{pmatrix} \dot{\boldsymbol{w}} \qquad [1.23]$$

where $\boldsymbol{D}_u$, $\boldsymbol{D}_c$ are partitions of the kinematic operator appearing in equation [1.3].

Dynamics

To derive the balance equations for the constrained problem, we will apply the VPP principle [1.7]. To express the virtual internal power, it is convenient to partition the stress $\boldsymbol{\sigma}(s,t)$ in two subsets, namely $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_c)^T$, where $\boldsymbol{\sigma}_u$ is an M_u-vector collecting the stresses dual of the unconstrained strains, and $\boldsymbol{\sigma}_c$ is an M_c-vector listing the stresses dual of the constrained strains. Thus, the VPP reads:

$$\int_{\mathcal{S}} (\boldsymbol{\sigma}_u^T \dot{\boldsymbol{\varepsilon}}_u + \boldsymbol{\sigma}_c^T \dot{\boldsymbol{\varepsilon}}_c) ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H$$
$$\forall (\dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}_u, \dot{\boldsymbol{\varepsilon}}_c) \,|\, (\dot{\boldsymbol{\varepsilon}}_u = \boldsymbol{D}_u \dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}_c = \mathbf{0} = \boldsymbol{D}_c \dot{\boldsymbol{w}}) \qquad [1.24]$$

where equations [1.23] have been accounted for. By using them in the internal power expression, we have:

$$\int_{\mathcal{S}} \boldsymbol{\sigma}_u^T \boldsymbol{D}_u \dot{\boldsymbol{w}} ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \quad \forall \dot{\boldsymbol{w}} | \boldsymbol{D}_c \dot{\boldsymbol{w}} = \mathbf{0}, \qquad [1.25]$$

which is still a *constrained* problem, since $\dot{\boldsymbol{w}}$ cannot be taken arbitrarily, but it must satisfy an auxiliary condition. By following the well-known *Lagrange multipliers* technique, we add to the previous equation the integral of $\boldsymbol{\lambda}^T \boldsymbol{D}_c \dot{\boldsymbol{w}} = 0$, where

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$\boldsymbol{\lambda} = \boldsymbol{\lambda}(s,t)$ are unknown Lagrangian multipliers, and rewrite equation [1.25] as an *unconstrained* problem[15]:

$$\int_{\mathcal{S}} \boldsymbol{\sigma}_u^T \boldsymbol{D}_u \dot{\boldsymbol{w}} ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H - \int_{\mathcal{S}} \boldsymbol{\lambda}^T \boldsymbol{D}_c \dot{\boldsymbol{w}} ds \quad \forall \dot{\boldsymbol{w}} \qquad [1.26]$$

But, if we rename $\boldsymbol{\lambda}$ as $\boldsymbol{\sigma}_c$, i.e., if we attribute to the constrained stresses the meaning of Lagrangian multipliers, this latter is equivalent to the original principle [1.24] with no constraints, i.e.:

$$\begin{gathered}\int_{\mathcal{S}} (\boldsymbol{\sigma}_u^T \dot{\boldsymbol{\varepsilon}}_u + \boldsymbol{\sigma}_c^T \dot{\boldsymbol{\varepsilon}}_c) ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \\ \forall (\dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}_u, \dot{\boldsymbol{\varepsilon}}_c) \,|\, (\dot{\boldsymbol{\varepsilon}}_u = \boldsymbol{D}_u \dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}_c = \boldsymbol{D}_c \dot{\boldsymbol{w}})\end{gathered} \qquad [1.27]$$

Therefore, after integration by parts, results identical to those supplied by the principle in equation [1.7] are recovered, but in split form; namely, the *split balance equations*:

$$\begin{pmatrix} \boldsymbol{D}_u^\star & \boldsymbol{D}_c^\star \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_u \\ \boldsymbol{\sigma}_c \end{pmatrix} = \boldsymbol{p} \qquad [1.28]$$

and the *split boundary conditions*:

$$\left[\dot{\boldsymbol{w}}^T \left(\begin{pmatrix} \boldsymbol{\mathcal{D}}_u^\star & \boldsymbol{\mathcal{D}}_c^\star \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_u \\ \boldsymbol{\sigma}_c \end{pmatrix} - \boldsymbol{P} \right) \right]_H = \boldsymbol{0} \qquad [1.29]$$

where $\boldsymbol{D}_u^\star$ and $\boldsymbol{D}_c^\star$ are the adjoint operators of $\boldsymbol{D}_u$ and $\boldsymbol{D}_c$.

Constitutive law

The left-hand member of equation [1.27] states that the internal virtual power $\mathcal{P}_{int}$ of a constrained system is made of two contributions: an *active* virtual power $\mathcal{P}_{act} := \int_{\mathcal{S}} \boldsymbol{\sigma}_u^T \dot{\boldsymbol{\varepsilon}}_u ds$ and a *reactive* virtual power $\mathcal{P}_{react} := \int_{\mathcal{S}} \boldsymbol{\sigma}_c^T \dot{\boldsymbol{\varepsilon}}_c ds$. Accordingly, $\boldsymbol{\sigma}_u$ is called the *active stress*, and $\boldsymbol{\sigma}_c$ the *reactive stress*. Note that, as suggested by the Lagrange multiplier technique, *the reactive stress spends zero virtual power in any admissible motion* (as reactive forces do in rigid-body mechanics), i.e. it satisfies the "perfect constraint postulate".

---

15. It is well-known, from the Variational Calculus, that the *constrained* problem $\delta I\left[u\left(s\right)\right] := \int_a^b \delta\mathcal{L}\left(u\left(s\right), u'\left(s\right)\right) ds = 0, \ \forall \delta f_i\left(u\left(s\right), u'\left(s\right)\right) = 0, \ i = 1, \ldots, n$ is equivalent to the *unconstrained* problem $\delta \tilde{I}\left[u\left(s\right), \lambda_i\left(s\right)\right] := \int_a^b \delta\mathcal{L} ds + \sum_{i=1}^n \int_a^b \lambda_i \delta f_i = 0$, where $\lambda_i\left(s\right)$ are Lagrange multipliers.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

We can still formulate a hyperelastic law for the active stresses, by requiring that the deformation work for unit length of the beam, i.e. the work $\frac{d}{ds}\left(\mathcal{P}_{act}dt\right)$ spent by the active stresses in time interval dt, equates the differential $d\phi$ of the elastic potential $\phi = \phi\left(\boldsymbol{\varepsilon}_u\right)$; from this, $\boldsymbol{\sigma}_u = \partial\phi/\partial\boldsymbol{\varepsilon}_u$ follows. If the potential is assumed quadratic, i.e. $\phi = 1/2\boldsymbol{\varepsilon}_u^T E_{uu}\boldsymbol{\varepsilon}_u$, the linear law follows:

$$\boldsymbol{\sigma}_u = \boldsymbol{E}_{uu}\boldsymbol{\varepsilon}_u \qquad [1.30]$$

General linear constraints

The constraints $\boldsymbol{\varepsilon}_c = 0$, so far considered, are probably so simple that they hide some interesting aspects of the problem. Therefore, we find useful a digression concerning more general kinematic constraints, of the kind:

$$\boldsymbol{B}\boldsymbol{\varepsilon} = \boldsymbol{0} \qquad [1.31]$$

where $\boldsymbol{B}$ is an $M_c \times M$ constant matrix. Of course, if $\boldsymbol{B} = [\boldsymbol{0},\ \boldsymbol{I}]$, the previous case is recovered. Later on in the book (Chapter 8), constraints like this will be addressed.

The VPP, with the constraint [1.31], reads:

$$\int_{\mathcal{S}} (\boldsymbol{\sigma}_a^T\dot{\boldsymbol{\varepsilon}} + \boldsymbol{\lambda}^T\boldsymbol{B}\dot{\boldsymbol{\varepsilon}})ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T\boldsymbol{p}ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T\boldsymbol{P}_H, \qquad \forall\left(\dot{\boldsymbol{w}}, \dot{\boldsymbol{\varepsilon}}\right)|\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}\dot{\boldsymbol{w}} \qquad [1.32]$$

where we denoted by $\boldsymbol{\sigma}_a$ the active stresses and by $\boldsymbol{\lambda}$ the Lagrangian multipliers. Note that, differently from the particular case examined previously, we did *not* introduce the constraint in the active part of the internal power. From the VPP, the balance equations are derived:

$$\boldsymbol{D}^\star\left(\boldsymbol{\sigma}_a + \boldsymbol{B}^T\boldsymbol{\lambda}\right) = \boldsymbol{0} \qquad [1.33]$$

together with the boundary conditions:

$$\left[\dot{\boldsymbol{w}}^T\left(\boldsymbol{\mathcal{D}}^\star\boldsymbol{\sigma} - \boldsymbol{P}\right)\right]_H = \boldsymbol{0} \qquad [1.34]$$

The internal virtual power states that the stress is the sum of an active and a passive quota, namely $\boldsymbol{\sigma} = \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_r$, with $\boldsymbol{\sigma}_r := \boldsymbol{B}^T\boldsymbol{\lambda}$. By assuming for the active stresses a linear elastic law, $\boldsymbol{\sigma}_a = \boldsymbol{E}\boldsymbol{\varepsilon}$, and taking into account the reactive part, the *elasto-reactive constitutive law* follows for the total stresses:

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon} + \boldsymbol{B}^T\boldsymbol{\lambda} \qquad [1.35]$$

This shows that, in general, each component of $\boldsymbol{\sigma}$ is partially active and partially reactive.

In the simplest constraint case, $\boldsymbol{\varepsilon}_c = \boldsymbol{0}$, for which $\boldsymbol{B} = [\boldsymbol{0},\ \boldsymbol{I}]$, the constitutive law [1.35] reads:

$$\begin{pmatrix}\boldsymbol{\sigma}_u\\ \boldsymbol{\sigma}_c\end{pmatrix} = \begin{pmatrix}\boldsymbol{E}_{uu} & \boldsymbol{E}_{uc}\\ \boldsymbol{E}_{cu} & \boldsymbol{E}_{cc}\end{pmatrix}\begin{pmatrix}\boldsymbol{\varepsilon}_u\\ \boldsymbol{0}\end{pmatrix} + \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{\lambda}\end{pmatrix} = \begin{pmatrix}\boldsymbol{E}_{uu}\boldsymbol{\varepsilon}_u\\ \boldsymbol{E}_{cu}\boldsymbol{\varepsilon}_u + \boldsymbol{\lambda}\end{pmatrix} \qquad [1.36]$$

so that $\boldsymbol{\sigma}_u$ is purely active, while $\boldsymbol{\sigma}_c$ is elasto-reactive. Since in equation [1.27] we zeroed $\dot{\boldsymbol{\varepsilon}}_c$ in the active internal power, the Lagrangian multiplier used in that equation accounts for both the active and reactive components of $\boldsymbol{\sigma}_c$. In the special (but frequent) case in which the elastic matrix $\boldsymbol{E}$ is diagonal, the constrained strains are *purely reactive*.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The constrained Fundamental Problem: the mixed formulation

By summarizing, the Fundamental Problem for the internally constrained beam is governed by: M_u strain–displacements relationships, with M_c constraints equations appended (equation [1.22]); N balance equation [1.28]; M_u purely elastic constitutive equation [1.30]; overall $2M_u + M_c + N$ equations. The unknowns involved are: M_u unconstrained strains $\boldsymbol{\varepsilon}_u$, $M_u + M_c$ stresses $(\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_c)$, N displacements $\boldsymbol{w}$, i.e. $2M_u + M_c + N$ unknowns. If we compare these numbers with that of the unconstrained problem, we note that M_c constrained strains $\boldsymbol{\varepsilon}_c$ disappeared, and also M_c constitutive elastic laws were canceled, this resulting in a contraction of the dimensions of the problem.

The fundamental equations cannot be combined according to the displacement method because the reactive stresses are independent of kinematic quantities. Therefore, a *mixed formulation* must be adopted, in terms of *both* displacements and reactions. Hence, the balance equations and constraints read (compare them with equations [1.18]):

$$\begin{aligned} &\boldsymbol{D}_u^\star \boldsymbol{E}_{uu} \boldsymbol{\mathcal{E}}_u + \boldsymbol{D}_c^\star \boldsymbol{\sigma}_c = \boldsymbol{p} \\ &\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w}, \boldsymbol{w}'\right) = \boldsymbol{0} \end{aligned} \qquad [1.37]$$

together with the boundary conditions:

$$\begin{aligned} &\boldsymbol{\mathcal{D}}_{uH}^\star \boldsymbol{E}_{uu} \boldsymbol{\mathcal{E}}_{uH} + \boldsymbol{\mathcal{D}}_{cH}^\star \boldsymbol{\sigma}_c = \boldsymbol{P}_H \\ &\boldsymbol{w}_H = \breve{\boldsymbol{w}}_H \end{aligned} \qquad [1.38]$$

of mechanical and geometric types, respectively (to be enforced alternatively). Equations [1.37] and [1.38] constitute a mixed boundary value problem, coupled in N displacements and M_c reactive stresses.

The linear theory

If equations [1.37] and [1.38a] are linearized around the trivial configuration, we have (compare them with equations [1.20] and [1.21]):

$$\begin{pmatrix} \boldsymbol{L}_u & \boldsymbol{D}_{0c}^\star \\ \boldsymbol{D}_{0c} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{\sigma}_c \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_0 \\ \boldsymbol{0} \end{pmatrix} \qquad [1.39]$$

with the mechanical boundary conditions:

$$\boldsymbol{\mathcal{L}}_{uH} \boldsymbol{w} + \boldsymbol{\mathcal{D}}_{0cH}^\star \boldsymbol{\sigma}_c = \boldsymbol{P}_{0H} \qquad [1.40]$$

where $\boldsymbol{L}_u, \boldsymbol{\mathcal{L}}_{uH}$ are *condensed* linear stiffness operators:

$$\boldsymbol{L}_u := \boldsymbol{D}_{0u}^\star \boldsymbol{E}_{uu} \boldsymbol{D}_{0u}, \qquad \boldsymbol{\mathcal{L}}_{uH} := \left[\boldsymbol{\mathcal{D}}_{0u}^\star \boldsymbol{E}_{uu} \boldsymbol{D}_{0u}\right]_H \qquad [1.41]$$

Note that $\boldsymbol{L}_u$ is self-adjoint.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

1.3.2 The displacement method for the internally constrained beam

The mixed formulation leads to balance equations that contain the reactive stresses. In order to formulate a problem purely in terms of displacements, as for the unconstrained beam, we have to eliminate the reactions. This goal could be reached, in principle, by performing linear algebraic/differential combinations among the original balance equation [1.28], by exploiting the fact that they are linear in the stresses. Thus, by using $M_c < N$ balance equations, we could eliminate as many reactive stresses, so obtaining $N_m := N - M_c$ equations in the active stresses only. The operation is called *condensation of the reactive stresses*. This circumstance is analogous to that of Lagrangian mechanics of constrained bodies, where one looks for the "Lagrange equations of motion", i.e. equations free of reactive forces.

As for rigid bodies, however, the condensation of the stresses via linear combination is neither simple nor convenient, but a variational (or integral, i.e. based on the VPP) approach is advised. This is based on a preliminary study of kinematics, in which $N_m = N - M_c$ displacements must be chosen as "master (or free) variables", and the remaining M_c "slave variables" related to them, in such a way to identically satisfy the M_c constraints. This operation represents a *condensation of the displacements*, dual to that of stresses, which balances the problem (M_c balance equations disappear, and M_c unknown slave displacements are eliminated). The master variables play a role identical to that of the Lagrangian parameters in rigid-body mechanics, i.e. they describe the most general configuration that is admissible with the constrains.

The true difficulty of the problem, however, consists of solving the constraint equations. Since they are *nonlinear* equations, they can rarely be tackled in the exact form, but, in contrast, a perturbation procedure must be applied, by resorting to series expansions. There is, however, another problem that makes the elimination of the variables difficult, due to the fact that the constraints are differential (and not algebraic!) equations[16]. In lucky cases in which a not-differentiated variable w_i appears in one equation, we can solve this equation (maybe, by means of a perturbation method) with respect to this variable, by using *algebraic* operations only, thus obtaining $w_i = f\left(w_j, w_j'\right)$ with $j \neq i$. If, in contrast, only first derivatives appear in the constraint equation, we could find $w_i' = f\left(w_j'\right)$ still using algebra. However, if w_i is needed, for example to evaluate inertia forces proportional to $\ddot{w}_i$, we should integrate, thus obtaining $w_i = \int f\left(w_j'\right) ds$, and also using geometric boundary conditions. In these circumstances, elimination of the variables could be inconvenient, and the mixed formulation would be preferable. Hybrid procedures, in

16. A similar circumstance occurs in Lagrangian mechanics, when *non-holonomic constraints* exist, which involve time-differentiated displacements.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

which only a sub-set of the variables is eliminated, are also possible, and relevant examples will be illustrated further on in the book (Chapter 4).

Condensation of displacements: master and slave variables

Let us assume that *all* the M_c constraint equations $\boldsymbol{\mathcal{E}}_c(\boldsymbol{w}, \boldsymbol{w}') = \mathbf{0}$ can be solved with respect to M_c *slave variables* $\boldsymbol{w}_s$, i.e. $\boldsymbol{w}_s = \boldsymbol{\mathcal{W}}_s(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots)$, where $\boldsymbol{w}_m$ are the remaining $N_m = N - M_c$ *master variables*; therefore, $\boldsymbol{w} := (\boldsymbol{w}_m, \boldsymbol{w}_s)^T = (\boldsymbol{w}_m, \boldsymbol{\mathcal{W}}_s(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots))^T$ or, in short[17]:

$$\boldsymbol{w} = \boldsymbol{\mathcal{W}}(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots) \qquad [1.42]$$

The generic configuration of the internally constrained beam is thus described by master configuration variables only. We will call this (nonlinear) relation the *constraint for displacements*, and we will say that the slave displacements have been condensed.

By time-differentiating the previous equation, we obtain the more general velocity field that is admissible for the instantaneous constraints, i.e.[18]:

$$\dot{\boldsymbol{w}} = \boldsymbol{A}(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots)\, \dot{\boldsymbol{w}}_m \qquad [1.43]$$

which, therefore, represents a (linear) *constraint for velocities*; in it, by omitting the arguments:

$$\boldsymbol{A} := \frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}_m} + \frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}'_m}\frac{\partial}{\partial s} + \ldots \qquad [1.44]$$

is a linear differential operator, represented by a $N \times N_m$ matrix: we will call it the *velocity constraint operator*. Since, from equation [1.23], it is $\boldsymbol{D}_c\dot{\boldsymbol{w}} = \mathbf{0}$, $\forall \boldsymbol{w}_m$, it follows from equation [1.43], that $\boldsymbol{D}_c\boldsymbol{A} = \mathbf{0}$.

With equations [1.42], and [1.43], the strain–displacements relationship (upper part of equation [1.22]) transforms into:

$$\boldsymbol{\varepsilon}_u = \boldsymbol{\mathcal{E}}_u(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots) \qquad [1.45]$$

to be sided by geometric boundary conditions:

$$\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad \boldsymbol{\mathcal{W}}_{sH}(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots) = \breve{\boldsymbol{w}}_{sH} \qquad [1.46]$$

17. The high-order derivatives of $\boldsymbol{w}_m$ have been denoted by ellipsis. As an example of the appearance of these terms, the equation $u_s - u'_m - u'^2_s = 0$ admits the solution $u_s = u'_m + u''^2_m + \text{h.o.t.}$

18. Note that $\boldsymbol{A} := (\boldsymbol{I}_m, \boldsymbol{A}_s)$, $\boldsymbol{I}_m$ being the identity matrix of order N_m.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where the know-terms can be freely imposed. Similarly, the strain-rate-velocity relationship (upper part of equation [1.23]) becomes:

$$\dot{\boldsymbol{\varepsilon}}_u = \boldsymbol{D}_u \boldsymbol{A} \dot{\boldsymbol{w}}_m \qquad [1.47]$$

where $\boldsymbol{D}_u = \boldsymbol{D}_u \left(\boldsymbol{\mathcal{W}} \left(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots \right), \boldsymbol{\mathcal{W}}' \left(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots \right) \right) = \boldsymbol{D}_u \left(\boldsymbol{w}_m, \boldsymbol{w}'_m, \ldots \right)$. We will refer to equations [1.45] and [1.47] as the *condensed kinematic relationships*.

REMARK 1.4. The condensation of the slave displacements leads to the appearance of the second- and high-order space-derivatives in the field, and first- and high-order space-derivatives at the boundaries.

Condensation of the balance equations

Now, we address the problem of condensation of the reactive stresses by the power balance approach. First, we rewrite the VPP in the form [1.25] we already used in the mixed formulation:

$$\int_{\mathcal{S}} \sigma_u^T \boldsymbol{D}_u \dot{\boldsymbol{w}} ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \boldsymbol{p} ds + \sum_{H=A}^{B} \dot{\boldsymbol{w}}_H^T \boldsymbol{P}_H \quad \forall \dot{\boldsymbol{w}} | \boldsymbol{D}_c \dot{\boldsymbol{w}} = \boldsymbol{0} \qquad [1.48]$$

Differently from that approach, however, we *will not introduce Lagrange multipliers* to express the geometrical constraints but, rather, we will use the master variables to identically satisfy them, via, $\dot{\boldsymbol{w}} = \boldsymbol{A}\dot{\boldsymbol{w}}_m$ (equation [1.43]). Since the reactive stresses do not appear in the VPP, the principle furnishes balance equations purely in the active stresses.

From a computational point of view, we find it conceptually clearer to reach the goal in two steps: (a) first, we integrate by parts equation [1.48] to free $\dot{\boldsymbol{w}}$ from the derivatives; and (b) then, we substitute the velocity constraint, and integrate again by parts, to free $\dot{\boldsymbol{w}}_m$ from the derivatives. By performing the first integration, we obtain:

$$\int_{\mathcal{S}} \dot{\boldsymbol{w}}^T \left(\boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p} \right) ds + \sum_{H=A}^{B} \left[\dot{\boldsymbol{w}}^T \left(\boldsymbol{\mathcal{D}}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{P} \right) \right]_H = 0 \quad \forall \dot{\boldsymbol{w}} | \boldsymbol{D}_c \dot{\boldsymbol{w}} = \boldsymbol{0} \quad [1.49]$$

where we used the extended Green identity [1.8] for the u-parts of the operators and stress[19]. Then, in order to satisfy the constraints, we express the velocities in terms of

19. Note that equation [1.49] could be directly obtained as a linear combinations of the split balance equations [1.28] and boundary conditions [1.29], *simply by ignoring the reactive stresses*. Therefore, if we already know the equilibrium equations for an unconstrained model, we can follow this shortcut, thus avoiding the first integration by parts.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

master velocities, namely $\dot{\boldsymbol{w}} = \boldsymbol{A}\dot{\boldsymbol{w}}_m$ in the field, and $\dot{\boldsymbol{w}} := (\dot{\boldsymbol{w}}_m, \dot{\boldsymbol{w}}_s)^T = \left(\dot{\boldsymbol{w}}_m, \dot{\boldsymbol{\mathcal{W}}}_s(\boldsymbol{w}_m, \ldots)\right)^T$ on the boundary, thus obtaining:

$$\int_{\mathcal{S}} \left(\boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p}\right)^T \boldsymbol{A}\dot{\boldsymbol{w}}_m ds + \sum_{H=A}^{B} \left[\dot{\boldsymbol{w}}_m^T \left(\boldsymbol{\mathcal{D}}_{um}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_m\right) + \dot{\boldsymbol{\mathcal{W}}}_s^T \left(\boldsymbol{\mathcal{D}}_{us}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_s\right)\right]_H = 0 \quad \forall \dot{\boldsymbol{w}}_m \tag{1.50}$$

where the partitions $\boldsymbol{\mathcal{D}}_u^\star := \left(\boldsymbol{\mathcal{D}}_{um}^\star, \boldsymbol{\mathcal{D}}_{us}^\star\right)^T$, $\boldsymbol{P} = (\boldsymbol{P}_m, \boldsymbol{P}_s)^T$ have been introduced[20].

Now, a second integration by parts is needed, involving the $\boldsymbol{A}$ operator, whose relevant extended green identity is written as[21]:

$$\int_{\mathcal{S}} \boldsymbol{p}_c^T \boldsymbol{A}\dot{\boldsymbol{w}}_m ds = \int_{\mathcal{S}} \dot{\boldsymbol{w}}_m^T \boldsymbol{A}^\star \boldsymbol{p}_c ds + \sum_{H=A}^{B} \left[\dot{\boldsymbol{w}}_m^T \boldsymbol{\mathcal{A}}^\star \boldsymbol{p}_c\right]_H \tag{1.51}$$

where $\boldsymbol{A}^\star$(of dimensions $N_m \times N$) is the adjoint of $\boldsymbol{A}$, and $\boldsymbol{\mathcal{A}}_H^\star$ (also of dimensions $N_m \times N$) is the associated operator at the boundary, to be referred to as *the equilibrium condensation operators*. By remembering the expression [1.44] for the velocity constraint operator, we get[22]:

$$\begin{aligned} \boldsymbol{A}^\star &:= \left(\frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}_m} - \frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}_m'}\right) - \frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}_m'}\frac{\partial}{\partial s} + \ldots\right)^T \\ \boldsymbol{\mathcal{A}}_H^\star &:= \mp\left(\frac{\partial \boldsymbol{\mathcal{W}}}{\partial \boldsymbol{w}_m'} + \ldots\right)_H^T \end{aligned} \tag{1.52}$$

With equation [1.51], the VPP [1.50] reads:

$$\begin{aligned} &\int_{\mathcal{S}} \dot{\boldsymbol{w}}_m^T \boldsymbol{A}^\star \left(\boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p}\right) ds + \sum_{H=A}^{B} \left[\dot{\boldsymbol{w}}_m^T \boldsymbol{\mathcal{A}}^\star \left(\boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p}\right)\right]_H \\ &+ \sum_{H=A}^{B} \left[\dot{\boldsymbol{w}}_m^T \left(\boldsymbol{\mathcal{D}}_{um}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_m\right) + \dot{\boldsymbol{\mathcal{W}}}_s^T \left(\boldsymbol{\mathcal{D}}_{us}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_s\right)\right]_H = 0 \qquad \forall \dot{\boldsymbol{w}}_m \end{aligned} \tag{1.53}$$

20. Note that $\dot{\boldsymbol{\mathcal{W}}}_s = \frac{\partial \boldsymbol{\mathcal{W}}_s}{\partial \boldsymbol{w}_m} \dot{\boldsymbol{w}}_m + \ldots$ also depends on $\dot{\boldsymbol{w}}_m$.

21. Here $\boldsymbol{p}_c$ is a dummy variable assuming the meaning of "external constraint force" $\boldsymbol{p}_c := \boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p} = \boldsymbol{D}_c^\star \boldsymbol{\sigma}_c$.

22. Note the analogy between equations [1.52] and [1.9].

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By taking into account that the velocities $\dot{\boldsymbol{w}}_m$ are arbitrary, the previous principle supplies the field equations:

$$\boldsymbol{A}^\star \boldsymbol{D}_u^\star \boldsymbol{\sigma}_u = \boldsymbol{A}^\star \boldsymbol{p} \qquad [1.54]$$

in which the two members represent Lagrange internal and external forces, respectively.

Consistently with the geometrical boundary conditions [1.46] (in which slave and master variables have been separated), the boundary terms in equation [1.53] also separate in:

$$\begin{aligned} &\left[\dot{\boldsymbol{w}}_m^T \left(\boldsymbol{\mathcal{A}}^\star \left(\boldsymbol{D}_u^\star \boldsymbol{\sigma}_u - \boldsymbol{p}\right) + \left(\boldsymbol{\mathcal{D}}_{um}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_m\right)\right)\right]_H = \mathbf{0} \\ &\left[\dot{\boldsymbol{\mathcal{W}}}_s^T \left(\boldsymbol{\mathcal{D}}_{us}^\star \boldsymbol{\sigma}_u - \boldsymbol{P}_s\right)\right]_H = \mathbf{0} \end{aligned} \qquad [1.55]$$

Equations [1.54] and [1.55] are the condensed equations sought for. The example of section 1.7 shows an application to a well-known linear problem of Timoshenko beam, with the purpose to corroborate the understanding of the theory illustrated here.

REMARK 1.5. The condensation of the reactive stresses leads to the appearance of space-derivatives of the loads, in the field equation [1.54] and to higher-order derivatives of the stresses in the equilibrium operator; moreover, it brings a contribution of the field load to the free boundaries (equation [1.55]).

REMARK 1.6. The condensed kinematic operator, $\boldsymbol{D}_u \boldsymbol{A}$ (equation [1.45]), and the condensed equilibrium operator, $\boldsymbol{A}^\star \boldsymbol{D}_u^\star$ (equation [1.54a], are mutually adjoint. Moreover, since $\boldsymbol{D}_c \boldsymbol{A} = \mathbf{0}$, then even $\boldsymbol{A}^\star \boldsymbol{D}_c^\star \equiv (\boldsymbol{D}_c \boldsymbol{A})^\star = \mathbf{0}$, thus explaining how $\boldsymbol{A}^\star$ annihilates $\boldsymbol{\sigma}_c$, and therefore the reactive stresses.

The constrained Fundamental Problem: the displacement formulation

By summarizing, the Fundamental Problem for the constrained beam, when formulated in terms of displacements $\boldsymbol{w}_m$ only, is governed by the following field equations:

– the condensed strain–displacement relationships [1.45];

– the condensed balance equations [1.54];

– the elastic law [1.30].

They are equipped with the alternative boundary conditions [1.55] and the geometric boundary condition [1.46]. By combining the field equations, we can express the balance equations in terms of the master displacements, namely:

$$\boldsymbol{A}^\star \boldsymbol{D}_u^\star \boldsymbol{E}_{uu} \boldsymbol{\mathcal{E}}_u = \boldsymbol{A}^\star \boldsymbol{p} \qquad [1.56]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The boundary conditions, when handled in the same way, read:

$$\begin{aligned}&[\boldsymbol{\mathcal{A}}^{\star}\boldsymbol{D}_u^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{\mathcal{D}}_{um}^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u]_H = [\boldsymbol{P}_m + \boldsymbol{\mathcal{A}}^{\star}\boldsymbol{p}]_H \\ &[\boldsymbol{\mathcal{D}}_{us}^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u]_H = [\boldsymbol{P}_s]_H \\ &\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad \boldsymbol{\mathcal{W}}_{sH}\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right) = \breve{\boldsymbol{w}}_{sH}\end{aligned} \tag{1.57}$$

The linear theory

If equations [1.56] and [1.57] are linearized around the reference configuration (and use is made of equation [1.47]), they read:

$$\boldsymbol{A}_0^{\star}\boldsymbol{L}_u\boldsymbol{A}_0\boldsymbol{w}_m = \boldsymbol{A}_0^{\star}\boldsymbol{p}_0 \tag{1.58}$$

together with:

$$\begin{aligned}&[\boldsymbol{\mathcal{A}}_0^{\star}\boldsymbol{L}_u\boldsymbol{A}_0 + \boldsymbol{\mathcal{L}}_{um}\boldsymbol{A}_0]_H\,\boldsymbol{w}_m = [\boldsymbol{P}_{0m} + \boldsymbol{\mathcal{A}}_0^{\star}\boldsymbol{p}_0]_H \\ &[\boldsymbol{\mathcal{L}}_{us}\boldsymbol{A}_0]_H\,\boldsymbol{w}_m = [\boldsymbol{P}_{0s}]_H \\ &\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad [\boldsymbol{A}_{0s}\boldsymbol{w}_m]_H = \breve{\boldsymbol{w}}_{sH}\end{aligned} \tag{1.59}$$

where the index 0 denotes evaluation at $\boldsymbol{w}_m = \boldsymbol{0}$, and moreover:

$$\begin{aligned}&\boldsymbol{L}_u := \boldsymbol{D}_{0u}^{\star}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}, \qquad \boldsymbol{\mathcal{L}}_{umH} := [\boldsymbol{\mathcal{D}}_{0um}^{\star}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}]_H \\ &\boldsymbol{\mathcal{L}}_{usH} := [\boldsymbol{\mathcal{D}}_{0us}^{\star}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}]_H\end{aligned} \tag{1.60}$$

are condensed linear elastic operators, in the domain and at the boundary, with $\boldsymbol{L}_u$ self-adjoint (remember equation [1.41]). In the last of the boundary conditions [1.59], $\boldsymbol{\mathcal{W}} = \boldsymbol{A}_0\boldsymbol{w}_m + \text{h.o.t.}$ has been used by exploiting equation [1.43] with the partition $\boldsymbol{A}_0 := (\boldsymbol{I}_m, \boldsymbol{A}_{0s})^T$.

Evaluation of the reactive stresses

Differently from the mixed formulation, in which the reactive stresses are included in the set of the primary unknowns, in the displacement formulations, they have to be evaluated by the balance equations, *after* the boundary value problem in the master displacements has been solved. First, the active stresses $\boldsymbol{\sigma}_u = \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u$ are evaluated, and then the split balance equations [1.28] and [1.29] written in the form:

$$\begin{aligned}&\boldsymbol{D}_c^{\star}\boldsymbol{\sigma}_c = \boldsymbol{p} - \boldsymbol{D}_u^{\star}\boldsymbol{\sigma}_u \\ &\boldsymbol{\mathcal{D}}_{cH}^{\star}\boldsymbol{\sigma}_c = \boldsymbol{P}_H - \boldsymbol{\mathcal{D}}_{uH}^{\star}\boldsymbol{\sigma}_u\end{aligned} \tag{1.61}$$

where $\boldsymbol{\sigma}_c$ are the unknowns. These equations represent the equilibrium of an ideal "rigid skeleton" of the beam, able to exert only reactive stresses $\boldsymbol{\sigma}_c$, under the action

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

of now *known* external and internal active forces. Such a problem, however, is overdetermined, since we have N equilibrium equations in $M_c < N$ unknowns. Therefore, in order it admits a solution, the knownn terms must satisfy a compatibility condition, i.e. *they must be orthogonal to all the solutions of the adjoint homogeneous problem*[23]. Since the latter is just $\boldsymbol{D}_c\dot{\boldsymbol{w}} = \boldsymbol{0}$, the know-term must be orthogonal to $\dot{\boldsymbol{w}} = \boldsymbol{A}\dot{\boldsymbol{w}}_m,\ \forall\dot{\boldsymbol{w}}_m$, and therefore compatibility requires[24]:

$$\int_{\mathcal{S}} \left(\boldsymbol{p} - \boldsymbol{D}_u^{\star}\boldsymbol{\sigma}_u\right)^T \boldsymbol{A}\dot{\boldsymbol{w}}_m ds + \sum_{H=A}^{B} \left[\left(\boldsymbol{P} - \boldsymbol{\mathcal{D}}_u^{\star}\boldsymbol{\sigma}_u\right)^T \boldsymbol{A}\dot{\boldsymbol{w}}_m\right]_H = 0 \qquad \forall\dot{\boldsymbol{w}}_m \quad [1.62]$$

This expresses that *the difference between the virtual powers spent by the active stresses and the external forces in any admissible velocity field is zero*. But this equation is exactly equation [1.50], which has been already satisfied in formulating the problem. Therefore, equations [1.61], although overdetermined, are integrable because they are not linearly independent (see, again, the example of section 1.7).

1.4 Internally unconstrained prestressed beams

Usually, as already observed, the reference configuration $\boldsymbol{\varepsilon} = \boldsymbol{0}$ is assumed to be stress-free. However, there exist problems in which it is more convenient to refer to a configuration in which the body is solicited by time-independent preloads, causing a state of prestress. Buckling falls in this class of problems; another set of problems in which prestress is important concerns strings and cables.

Of course, a prestressed beam is also prestrained, this entailing a change of geometry with respect to its natural state. Thus, for example, if the prestress of straight beam is caused by an axial load, the beam is shortened with respect to its natural length; if the beam is transversely loaded, it is bent, twisted and shear-strained; if, for example, the beam has an initial uniform curvature, this is rendered non-uniform by preloads. Strictly speaking, such changes of geometry should be accounted for in analyzing the mechanical behavior of the beam, when incremental loads, additional with respect to preloads, are applied. Such an approach, however, would nullify the advantage to refer to a prestressed configuration because the relevant geometry would be more complex than the natural geometry. Therefore, in order to simplify the problem, *prestrains and deformations produced by preloads are neglected*, so that the reference prestressed configuration is confused with the

23. This property, known in functional analysis (Fredholm alternative), can be considered as a straightforward extension of the Rouché–Capelli theorem, holding in Algebra.

24. Note that we did not account for *any* geometric boundary condition in evaluating $\dot{\boldsymbol{w}}_m$, in order to make the treatment as general as possible.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

natural configuration. In other words, the geometrical effects caused by the prestress are ignored.

1.4.1 The nonlinear theory

With these ideas in mind, let us consider a beam under time-independent *preloads* $\mathring{\boldsymbol{p}}(\boldsymbol{w})$, $\mathring{\boldsymbol{P}}_H(\boldsymbol{w})$, possibly dependent on the configuration, equilibrated with *prestresses* $\mathring{\boldsymbol{\sigma}}(s)$, and assume the equilibrium configuration as a *known* reference configuration. As a result, the following equations hold:

$$\begin{aligned} \boldsymbol{D}_0^\star \mathring{\boldsymbol{\sigma}} &= \mathring{\boldsymbol{p}}_0 \\ \boldsymbol{\mathcal{D}}_{0H}^\star \mathring{\boldsymbol{\sigma}} &= \mathring{\boldsymbol{P}}_{0H} \end{aligned} \qquad [1.63]$$

where the equilibrium operators and the loads have been evaluated at $\boldsymbol{w} \equiv \boldsymbol{0}$. Let us assume, then, that at time $t = 0$, *incremental loads* $\tilde{\boldsymbol{p}}(\boldsymbol{w})$, $\tilde{\boldsymbol{P}}_H(\boldsymbol{w})$ are applied to the beam. These loads bring the beam to occupy a new (possibly time-dependent) current configuration, described by the generalized displacements $\boldsymbol{w}(s,t)$, measured with respect to the reference configuration. Accordingly, kinematics is still governed by equations [1.1] and [1.2], that we repeat here:

$$\begin{aligned} \boldsymbol{\varepsilon} &= \boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}') \\ \boldsymbol{w}_H &= \breve{\boldsymbol{w}}_H \end{aligned} \qquad [1.64]$$

Similarly, the balance equations are still equations [1.11] and [1.12], but with *total loads* applied, accounting for the current values of the preloads:

$$\begin{aligned} \boldsymbol{D}^\star(\boldsymbol{w}, \boldsymbol{w}')\,\boldsymbol{\sigma} &= \mathring{\boldsymbol{p}}(\boldsymbol{w}) + \tilde{\boldsymbol{p}}(\boldsymbol{w}) \\ \boldsymbol{\mathcal{D}}_H^\star(\boldsymbol{w}, \boldsymbol{w}')\,\boldsymbol{\sigma} &= \mathring{\boldsymbol{P}}_H(\boldsymbol{w}) + \tilde{\boldsymbol{P}}_H(\boldsymbol{w}) \end{aligned} \qquad [1.65]$$

Concerning the elastic law, we have to modify the Hook law in order to get $\boldsymbol{\sigma} = \mathring{\boldsymbol{\sigma}}$ when $\boldsymbol{\varepsilon} = \boldsymbol{0}$. This is accomplished by considering an elastic potential represented by a *complete* quadratic polynomial[25]:

$$\phi(\boldsymbol{\varepsilon}) = \phi_0 + \mathring{\boldsymbol{\sigma}}^T \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon}^T \boldsymbol{E} \boldsymbol{\varepsilon} \qquad [1.66]$$

Then, a linear non-homogeneous law follows from equation [1.14]:

$$\boldsymbol{\sigma} = \mathring{\boldsymbol{\sigma}} + \boldsymbol{E}\boldsymbol{\varepsilon} \qquad [1.67]$$

25. The constant ϕ_0 is unessential, since a potential function is always defined to within a constant. Therefore, we will omit it further on.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The governing equations can be combined according to the displacement method, as done for the stress-free beam. The stress–displacement relationships then read:

$$\boldsymbol{\sigma} = \mathring{\boldsymbol{\sigma}} + \boldsymbol{E}\boldsymbol{\mathcal{E}}(\boldsymbol{w}, \boldsymbol{w}') \quad [1.68]$$

and, as a result, the balance equations and the mechanical boundary conditions transform into:

$$\begin{aligned} &\boldsymbol{D}^\star \boldsymbol{E}\boldsymbol{\mathcal{E}} + (\boldsymbol{D}^\star \mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{p}}) = \tilde{\boldsymbol{p}} \\ &\boldsymbol{\mathcal{D}}_H^\star \boldsymbol{E}\boldsymbol{\mathcal{E}}_H + \left(\boldsymbol{\mathcal{D}}_H^\star \mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{P}}_H\right) = \tilde{\boldsymbol{P}}_H \end{aligned} \quad [1.69]$$

Equations [1.69] and the geometric boundary conditions [1.64b] constitute a nonlinear boundary value problem for the main unknowns $\boldsymbol{w}$.

REMARK 1.7. Equations [1.69] state that the incremental loads $\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{P}}_H$ are equilibrated not only by the incremental elastic forces, as happens in stress-free beams, but also by the imbalance between preloads and prestresses, which is *caused by the change of geometry*.

1.4.2 The linearized theory

Very often, in buckling problems, we are interested in determining the critical load only, or the response of the beam to *small* incremental loads, acting as disturbances/imperfections of the prestressed equilibrium configuration, mostly when the beam is close to the bifurcation. Similarly, in dynamics, we want to evaluate the frequencies of a prestressed beam or cable, or the response of the structure when *small* incremental loads externally excite the beam, especially when this is close to the resonance. In all these cases, the linearized version of equation [1.69] is sufficient to give an accurate response (i.e. to within the effects of the neglected prestrains), leading to a differential eigenvalue problem (for critical load or frequencies) or a non-homogeneous boundary value problem (for small incremental loads). The relevant framework is called *Linearized Theory*[26].

26. We use the wordings *linearized* theory for prestressed beams, and *linear* theory for stress-free beams.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

To linearize equations [1.69], we have to move one order ahead with respect to the series expansions [1.19][27]. Concerning the field equations, we have[28]:

$$\begin{aligned}
&\boldsymbol{D}^{\star}\left(\boldsymbol{w}, \boldsymbol{w}'\right) \mathring{\boldsymbol{\sigma}} = \boldsymbol{D}_0^{\star} \mathring{\boldsymbol{\sigma}} + \left(\frac{\partial\left(\boldsymbol{D}^{\star} \mathring{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{w}}\right)_0 \boldsymbol{w} + \left(\frac{\partial\left(\boldsymbol{D}^{\star} \mathring{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{w}'}\right)_0 \boldsymbol{w}' + \text{h.o.t.} \\
&\mathring{\boldsymbol{p}}(\boldsymbol{w}) = \mathring{\boldsymbol{p}}_0 + \left(\frac{\partial \mathring{\boldsymbol{p}}}{\partial \boldsymbol{w}}\right)_0 \boldsymbol{w} + \text{h.o.t.}, \qquad \tilde{\boldsymbol{p}}(\boldsymbol{w}) = \tilde{\boldsymbol{p}}_0 + \text{h.o.t.}
\end{aligned} \qquad [1.70]$$

in which we assumed $\tilde{\boldsymbol{p}}(\boldsymbol{w})$ and $\boldsymbol{w}$ small of the same order. Similarly, for the boundary conditions, we have:

$$\begin{aligned}
&\boldsymbol{\mathcal{D}}_H^{\star}\left(\boldsymbol{w}, \boldsymbol{w}'\right) \mathring{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{D}}_{0H}^{\star} \mathring{\boldsymbol{\sigma}} + \left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_H^{\star} \mathring{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{w}}\right)_0 \boldsymbol{w} + \left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_H^{\star} \mathring{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{w}'}\right)_0 \boldsymbol{w}' + \text{h.o.t.} \\
&\mathring{\boldsymbol{P}}_H(\boldsymbol{w}) = \mathring{\boldsymbol{P}}_{0H} + \left(\frac{\partial \mathring{\boldsymbol{P}}_H}{\partial \boldsymbol{w}}\right)_0 \boldsymbol{w} + \text{h.o.t.}, \qquad \tilde{\boldsymbol{P}}_H(\boldsymbol{w}) = \tilde{\boldsymbol{P}}_{0H} + \text{h.o.t.}
\end{aligned} \qquad [1.71]$$

By retaining first-order terms only in the series expansions, and accounting for the equilibrium conditions [1.63] of the prestressed configuration, we obtain:

$$\begin{aligned}
&\boldsymbol{L}\boldsymbol{w} + \boldsymbol{G}\boldsymbol{w} = \tilde{\boldsymbol{p}}_0 \\
&\boldsymbol{\mathcal{L}}_H \boldsymbol{w} + \boldsymbol{\mathcal{G}}_H \boldsymbol{w} = \tilde{\boldsymbol{P}}_0
\end{aligned} \qquad [1.72]$$

where $\boldsymbol{L}$ and $\boldsymbol{\mathcal{L}}_H$ are the already introduced elastic stiffness operators of the linear theory (equation [1.21]), and:

$$\begin{aligned}
&\boldsymbol{G} := \left(\frac{\partial\left(\boldsymbol{D}^{\star} \mathring{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial\left(\boldsymbol{D}^{\star} \mathring{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s} - \left(\frac{\partial \mathring{\boldsymbol{p}}}{\partial \boldsymbol{w}}\right)_0 \\
&\boldsymbol{\mathcal{G}}_H := \left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_H^{\star} \mathring{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_H^{\star} \mathring{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s} - \left(\frac{\partial \mathring{\boldsymbol{P}}_H}{\partial \boldsymbol{w}}\right)_0
\end{aligned} \qquad [1.73]$$

are *geometric stiffness operators*, in the domain and on the boundary, respectively, accounting for prestress.

REMARK 1.8. The geometric stiffness accounts for the effect on the equilibrium of an *infinitely small change of geometry* of the beam, when it passes from the reference to an *adjacent* current configuration. The imbalance between prestresses and preloads, $\boldsymbol{D}^{\star}\left(\boldsymbol{w}, \boldsymbol{w}'\right) \mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{p}}\left(\boldsymbol{w}\right)$, when linearized, is just $\boldsymbol{G}\boldsymbol{w}$.

27. From this circumstance, the linearized theory is also called the "second-order theory", this being a less precise wording, often used in technical circles.

28. Although $\mathring{\boldsymbol{\sigma}}$ is independent of $\boldsymbol{w}$, we prefer to differentiate the product $(\boldsymbol{D}^{\star} \mathring{\boldsymbol{\sigma}})$, to remember that $\boldsymbol{D}^{\star}$ operates on $\mathring{\boldsymbol{\sigma}}$, and, moreover, to avoid introducing the derivative of a matrix with respect to a vector.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The geometric stiffness operator

To obtain an expression for the geometric stiffness operator [1.73a] in terms of strains, we use equation [1.9a] and obtain[29]:

$$\begin{aligned}\boldsymbol{D}^{\star}\mathring{\boldsymbol{\sigma}} &= \left(\frac{\partial \boldsymbol{\mathcal{E}}}{\partial \boldsymbol{w}}\right)^{T}\mathring{\boldsymbol{\sigma}} - \frac{\partial}{\partial s}\left(\frac{\partial \boldsymbol{\mathcal{E}}}{\partial \boldsymbol{w}'}\right)^{T}\mathring{\boldsymbol{\sigma}} - \left(\frac{\partial \boldsymbol{\mathcal{E}}}{\partial \boldsymbol{w}'}\right)^{T}\mathring{\boldsymbol{\sigma}}' \\ &= \sum_{i=1}^{M}\left(\left(\frac{\partial \mathcal{E}_i}{\partial \boldsymbol{w}} - \frac{\partial}{\partial s}\frac{\partial \mathcal{E}_i}{\partial \boldsymbol{w}'}\right)\mathring{\sigma}_i - \frac{\partial \mathcal{E}_i}{\partial \boldsymbol{w}'}\mathring{\sigma}_i'\right)\end{aligned} \qquad [1.74]$$

From this, we can evaluate the contributions to $\boldsymbol{G}\boldsymbol{w}$, namely:

$$\begin{aligned}\left(\frac{\partial\left(\boldsymbol{D}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{w}}\right)_0 \boldsymbol{w} &= \sum_{i=1}^{M}\left(\left(\boldsymbol{A}_i\boldsymbol{w} - \boldsymbol{B}_i^T\boldsymbol{w}'\right)\mathring{\sigma}_i - \boldsymbol{B}_i^T\boldsymbol{w}\mathring{\sigma}_i'\right) \\ \left(\frac{\partial\left(\boldsymbol{D}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{w}'}\right)_0 \boldsymbol{w}' &= \sum_{i=1}^{M}\left(\left(\boldsymbol{B}_i\boldsymbol{w}' - \boldsymbol{C}_i\boldsymbol{w}''\right)\mathring{\sigma}_i - \boldsymbol{C}_i\boldsymbol{w}'\mathring{\sigma}_i'\right)\end{aligned} \qquad [1.75]$$

where the following matrices of the second derivatives of $\mathcal{E}_i$, evaluated at $(\boldsymbol{w}, \boldsymbol{w}') = (\boldsymbol{0}, \boldsymbol{0})$, have been introduced[30]:

$$\boldsymbol{A}_i := \left(\frac{\partial^2 \mathcal{E}_i}{\partial \boldsymbol{w}^2}\right)_0, \qquad \boldsymbol{B}_i := \left(\frac{\partial^2 \mathcal{E}_i}{\partial \boldsymbol{w}\partial \boldsymbol{w}'}\right)_0, \qquad \boldsymbol{C}_i := \left(\frac{\partial^2 \mathcal{E}_i}{\partial \boldsymbol{w}'^2}\right)_0 \qquad [1.77]$$

being $\boldsymbol{A}_i = \boldsymbol{A}_i^T$, $\boldsymbol{C}_i = \boldsymbol{C}_i^T$, while $\boldsymbol{B}_i^T = \left(\frac{\partial^2 \mathcal{E}_i}{\partial \boldsymbol{w}'\partial \boldsymbol{w}}\right)_0 \neq \boldsymbol{B}_i$[31]. Hence:

$$\boldsymbol{G} = \sum_{i=1}^{M}\left(\left(\boldsymbol{A}_i\mathring{\sigma}_i - \boldsymbol{B}_i^T\mathring{\sigma}_i'\right) + \left(\boldsymbol{B}_i\mathring{\sigma}_i - \boldsymbol{B}_i^T\mathring{\sigma}_i - \boldsymbol{C}_i\mathring{\sigma}_i'\right)\frac{\partial}{\partial s} - \boldsymbol{C}_i\mathring{\sigma}_i\frac{\partial^2}{\partial s^2}\right) - \left(\frac{\partial \mathring{\boldsymbol{p}}}{\partial \boldsymbol{w}}\right)_0 \qquad [1.78]$$

The stiffness operator at the boundary can be obtained in a similar manner. By using equation [1.9] we have:

$$\boldsymbol{\mathcal{D}}_H^{\star}\mathring{\boldsymbol{\sigma}} := \mp\left(\frac{\partial \boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right)}{\partial \boldsymbol{w}'}\right)_H^T \mathring{\boldsymbol{\sigma}} \qquad [1.79]$$

---

29. Note that, e.g., $\left(\frac{\partial \boldsymbol{\mathcal{E}}}{\partial \boldsymbol{w}}\right)^T$ is the column-wise matrix $\left[\frac{\partial \mathcal{E}_1}{\partial \boldsymbol{w}}, \frac{\partial \mathcal{E}_2}{\partial \boldsymbol{w}}, \ldots, \frac{\partial \mathcal{E}_M}{\partial \boldsymbol{w}}\right]$, where the derivative of a scalar with respect to a vector denotes a column vector.

30. Note that these are sub-matrices of the Hessian of $\mathcal{E}_i$ at the origin, once the variables have been ordered as $(\boldsymbol{w}, \boldsymbol{w}')$:

$$\boldsymbol{H}_i^0 = \begin{pmatrix}\boldsymbol{A}_i & \boldsymbol{B}_i \\ \boldsymbol{B}_i^T & \boldsymbol{C}_i\end{pmatrix} \qquad [1.76]$$

31. Indeed, $\boldsymbol{B}_i = \left[\frac{\partial^2 \mathcal{E}_i}{\partial u_j \partial u_k'}\right]_0$, $\boldsymbol{B}_i^T = \left[\frac{\partial^2 \mathcal{E}_i}{\partial u_j' \partial u_k}\right]_0$ with j being the row and k the column.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

from which:

$$\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}}\right)_{0H}\boldsymbol{w}=\mp\sum_{i=1}^{M}\boldsymbol{B}_i^T\boldsymbol{w}\mathring{\sigma}_i,\qquad\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}^{\star}\mathring{\boldsymbol{\sigma}}\right)}{\partial\boldsymbol{w}'}\right)_{0H}\boldsymbol{w}'=\mp\sum_{i=1}^{M}\boldsymbol{C}_i\boldsymbol{w}'\mathring{\sigma}_i' \quad [1.80]$$

and therefore, from equation [1.73b], we finally get:

$$\boldsymbol{\mathcal{G}}_H=\mp\sum_{i=1}^{M}\left(\boldsymbol{B}_i^T\mathring{\sigma}_i+\boldsymbol{C}_i\mathring{\sigma}_i\frac{\partial}{\partial s}\right)-\left(\frac{\partial\mathring{\boldsymbol{P}}}{\partial\boldsymbol{w}}\right)_{0H} \quad [1.81]$$

1.5 Internally constrained prestressed beams

We consider again a prestressed beam, but refer ourselves to an internally constrained model, so that all the aspects illustrated in the previous sections are involved in this more complex problem. As for the stress-free beam, we want to tackle both the mixed and displacement formulations, and as for the prestressed beam, we want to develop models in the nonlinear and linearized frameworks. Therefore, four different models are illustrated further on.

1.5.1 The nonlinear mixed formulation

Let us assume that the beam is in equilibrium under time-independent but configuration-dependent preloads $\mathring{\boldsymbol{p}}(\boldsymbol{w})$, $\mathring{\boldsymbol{P}}_H(\boldsymbol{w})$, and prestresses $\mathring{\boldsymbol{\sigma}}$, and ignore any deformation of the beam, so that the equilibrium configuration is confused with the reference configuration. Prestresses can be either of active or reactive type, i.e. $\mathring{\boldsymbol{\sigma}}=(\mathring{\boldsymbol{\sigma}}_u,\mathring{\boldsymbol{\sigma}}_c)^T$, and we assume they have been already determined from a prestress analysis. As a result, the equilibrium equations [1.63] hold, with the partition introduced:

$$\begin{aligned}\boldsymbol{D}_{0u}^{\star}\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{D}_{0c}^{\star}\mathring{\boldsymbol{\sigma}}_c&=\mathring{\boldsymbol{p}}_0\\ \boldsymbol{\mathcal{D}}_{0uH}^{\star}\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{\mathcal{D}}_{0cH}^{\star}\mathring{\boldsymbol{\sigma}}_c&=\mathring{\boldsymbol{P}}_{0H}\end{aligned} \quad [1.82]$$

and where the index 0 denotes evaluation at the trivial configuration.

Let us consider, then, incremental loads $\tilde{\boldsymbol{p}}(\boldsymbol{w})$, $\tilde{\boldsymbol{P}}_H(\boldsymbol{w})$ applied to the beam at $t=0$. They cause the beam to assume a current unknown configuration, in which unconstrained and constrained strains are related to displacements by equations [1.22], which we repeat here:

$$\begin{aligned}\boldsymbol{\varepsilon}_u&=\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)\\ \boldsymbol{0}&=\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w},\boldsymbol{w}'\right)\end{aligned} \quad [1.83]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The balance equations [1.65], relevant to the current configuration, in the split form read:

$$\begin{aligned} &\boldsymbol{D}_u^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{\sigma}_u+\boldsymbol{D}_c^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{\sigma}_c=\mathring{\boldsymbol{p}}(\boldsymbol{w})+\tilde{\boldsymbol{p}}(\boldsymbol{w}) \\ &\boldsymbol{\mathcal{D}}_{uH}^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{\sigma}_a+\boldsymbol{\mathcal{D}}_{cH}^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{\sigma}_c=\mathring{\boldsymbol{P}}_H(\boldsymbol{w})+\tilde{\boldsymbol{P}}_H(\boldsymbol{w}) \end{aligned} \qquad [1.84]$$

The constitutive law is non-homogeneous, as equation [1.67], but it concerns only the active stresses, as equation [1.30], namely:

$$\boldsymbol{\sigma}_u=\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{E}_{uu}\boldsymbol{\varepsilon}_u \qquad [1.85]$$

By using the previous equations, we write the balance equations and the mechanical boundary conditions in terms of displacements and incremental reactive stresses:

$$\tilde{\boldsymbol{\sigma}}_c:=\boldsymbol{\sigma}_c-\mathring{\boldsymbol{\sigma}}_c \qquad [1.86]$$

Moreover, we append to them the constraints and the geometric boundary conditions. Hence, the final boundary value problem consists of the following field equations:

$$\begin{aligned} &\boldsymbol{D}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\boldsymbol{D}_c^\star\tilde{\boldsymbol{\sigma}}_c+\left(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}}-\mathring{\boldsymbol{p}}\right)=\tilde{\boldsymbol{p}} \\ &\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{0} \end{aligned} \qquad [1.87]$$

and the boundary conditions[32]:

$$\begin{aligned} &\boldsymbol{\mathcal{D}}_{uH}^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_{uH}+\boldsymbol{\mathcal{D}}_{cH}^\star\tilde{\boldsymbol{\sigma}}_c+\left(\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}}-\mathring{\boldsymbol{P}}_H\right)=\tilde{\boldsymbol{P}}_H \\ &\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H \end{aligned} \qquad [1.88]$$

Comparison with equations [1.37] and [1.38], relevant to the unprestressed beam, highlights the contribution of the prestress.

1.5.2 The linearized mixed formulation

To linearize equations [1.87a] and [1.88a], we use the series expansions [1.70] and [1.71] and assume incremental reactive stresses and loads to be small first-order quantities; moreover, we replace the constraint condition by its first-order approximation. Thus, we get the linearized equations (compare them with

32. Note that the prestresses have been merged, via $\boldsymbol{D}_u^\star\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{D}_c^\star\mathring{\boldsymbol{\sigma}}_c=\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}}$ and $\boldsymbol{\mathcal{D}}_{uH}^\star\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{\mathcal{D}}_{cH}^\star\mathring{\boldsymbol{\sigma}}_c=\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}}$, since they are known terms in this analysis.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

equations [1.39] and [1.40], relevant to the linear theory of the unprestressed beam, and with equations [1.72], relevant to the linearized theory of the prestressed unconstrained beam):

$$\left(\begin{pmatrix}\boldsymbol{L}_u & \boldsymbol{D}_{0c}^{\star}\\ \boldsymbol{D}_{0c} & \mathbf{0}\end{pmatrix}+\begin{pmatrix}\boldsymbol{G} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\right)\begin{pmatrix}\boldsymbol{w}\\ \tilde{\boldsymbol{\sigma}}_c\end{pmatrix}=\begin{pmatrix}\tilde{\boldsymbol{p}}_0\\ \mathbf{0}\end{pmatrix} \qquad [1.89]$$

with the mechanical boundary conditions:

$$\boldsymbol{\mathcal{L}}_{uH}\boldsymbol{w}+\boldsymbol{\mathcal{G}}_H\boldsymbol{w}+\boldsymbol{\mathcal{D}}_{0cH}^{\star}\tilde{\boldsymbol{\sigma}}_c=\tilde{\boldsymbol{P}}_{0H} \qquad [1.90]$$

where $\boldsymbol{L}_u,\ \boldsymbol{\mathcal{L}}_{uH}$ are defined in equations [1.41], and $\boldsymbol{G},\ \boldsymbol{\mathcal{G}}_H$ in equations [1.73].

1.5.3 The nonlinear displacement formulation

If we follow the displacement formulation, we have to condense slave displacements and reactive stresses, as we did for the stress-free beam. Kinematics is governed by the condensed strain–displacement relationships [1.45] and geometric boundary conditions [1.46], i.e.:

$$\begin{aligned}\boldsymbol{\varepsilon}_u &= \boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)\\ \boldsymbol{w}_{mH} &= \breve{\boldsymbol{w}}_{mH},\quad \boldsymbol{\mathcal{W}}_{sH}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)=\breve{\boldsymbol{w}}_{sH}\end{aligned} \qquad [1.91]$$

Equilibrium is governed by equations [1.54] and boundary conditions by equations [1.55], provided total loads $\boldsymbol{p}:=\mathring{\boldsymbol{p}}+\tilde{\boldsymbol{p}},\ \boldsymbol{P}_H:=\mathring{\boldsymbol{P}}_H+\tilde{\boldsymbol{P}}_H$ are taken into account, i.e.:

$$\boldsymbol{A}^{\star}\boldsymbol{D}_u^{\star}\boldsymbol{\sigma}_u=\boldsymbol{A}^{\star}\left(\mathring{\boldsymbol{p}}+\tilde{\boldsymbol{p}}\right) \qquad [1.92]$$

and:

$$\begin{aligned}&\left[\left(\boldsymbol{\mathcal{A}}^{\star}\left(\boldsymbol{D}_u^{\star}\boldsymbol{\sigma}_u-\mathring{\boldsymbol{p}}-\tilde{\boldsymbol{p}}\right)+\left(\boldsymbol{\mathcal{D}}_{um}^{\star}\boldsymbol{\sigma}_u-\mathring{\boldsymbol{P}}_m-\tilde{\boldsymbol{P}}_m\right)\right)\right]_H=\mathbf{0}\\ &\left[\left(\boldsymbol{\mathcal{D}}_{us}^{\star}\boldsymbol{\sigma}_u-\mathring{\boldsymbol{P}}_s-\tilde{\boldsymbol{P}}_s\right)\right]_H=\mathbf{0}\end{aligned} \qquad [1.93]$$

where all the operators and loads depend on $\boldsymbol{w}_m$ and its derivatives, via $\boldsymbol{w}=\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)$ (equation [1.42]). Finally, the constitutive law is given by equation [1.85], i.e.:

$$\boldsymbol{\sigma}_u=\mathring{\boldsymbol{\sigma}}_u+\boldsymbol{E}_{uu}\boldsymbol{\varepsilon}_u \qquad [1.94]$$

Combination of the previous field equations leads to (compare them with equation [1.56], where prestress was absent):

$$\boldsymbol{A}^{\star}\boldsymbol{D}_u^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\boldsymbol{A}^{\star}\left(\boldsymbol{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u-\mathring{\boldsymbol{p}}\right)=\boldsymbol{A}^{\star}\tilde{\boldsymbol{p}} \qquad [1.95]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and the relevant boundary conditions (compare with [1.57]):

$$\begin{aligned}
&\left[\mathcal{A}^{\star}\boldsymbol{D}_u^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\boldsymbol{\mathcal{D}}_{um}^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\mathcal{A}^{\star}\left(\boldsymbol{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u-\mathring{\boldsymbol{p}}\right)+\left(\boldsymbol{\mathcal{D}}_{um}^{\star}\mathring{\boldsymbol{\sigma}}_u-\mathring{\boldsymbol{P}}_m\right)\right]_H\\
&=\left[\tilde{\boldsymbol{P}}_m+\mathcal{A}^{\star}\tilde{\boldsymbol{p}}\right]_H\\
&\left[\boldsymbol{\mathcal{D}}_{us}^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\left(\boldsymbol{\mathcal{D}}_{us}^{\star}\mathring{\boldsymbol{\sigma}}_u-\mathring{\boldsymbol{P}}_s\right)\right]_H=\left[\tilde{\boldsymbol{P}}_s\right]_H\\
&\boldsymbol{w}_{mH}=\breve{\boldsymbol{w}}_{mH},\quad \boldsymbol{\mathcal{W}}_{sH}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)=\breve{\boldsymbol{w}}_{sH}
\end{aligned}\qquad [1.96]$$

Once the problem has been solved, the total reactive stresses follow from the not-condensed equilibrium equations.

REMARK 1.9. Equations [1.95] and [1.96], as equations [1.69], contain unbalanced preloads–prestress forces. These, however, differently from the previous formulations, are expressed in terms of active forces only, since premultiplication by $\boldsymbol{A}^{\star}$ filters the reactive contributions.

1.5.4 The linearized displacement formulation

Linearization of equations [1.95] and [1.96] calls for using series expansions of all operators and loads. Elastic terms and incremental loads can been dealt with as for the unprestressed beam (equations [1.58] and [1.59]). Additional geometric terms arise from imbalanced prestresses and preloads, requiring expansion of the u-parts of the equilibrium operators,$\boldsymbol{D}_u^{\star}$, $\boldsymbol{\mathcal{D}}_u^{\star}=\left(\boldsymbol{\mathcal{D}}_{um}^{\star},\boldsymbol{\mathcal{D}}_{us}^{\star}\right)^T$ and of the preloads $\mathring{\boldsymbol{p}}$. Now, as we observed, these quantities depend on the master displacements via the constraints, e.g. $\boldsymbol{D}_u^{\star}=\boldsymbol{D}_u^{\star}\left(\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right),\boldsymbol{\mathcal{W}}'\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)\right)$; thus, we find it more convenient to first expand them with respect to $\boldsymbol{w},\boldsymbol{w}'$, and then to use $\boldsymbol{w}=\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)=\boldsymbol{A}_0\boldsymbol{w}_m+\text{h.o.t.}$, namely:

$$\begin{aligned}
\boldsymbol{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u&=\boldsymbol{D}_{0u}^{\star}\mathring{\boldsymbol{\sigma}}_u+\left(\frac{\partial\left(\boldsymbol{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u\right)}{\partial\boldsymbol{w}}\right)_0\boldsymbol{w}+\left(\frac{\partial\left(\boldsymbol{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u\right)}{\partial\boldsymbol{w}'}\right)_0\boldsymbol{w}'+\text{h.o.t.}\\
&=\boldsymbol{D}_{0u}^{\star}\mathring{\boldsymbol{\sigma}}_u+\left[\left(\frac{\partial\left(\boldsymbol{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u\right)}{\partial\boldsymbol{w}}\right)_0+\left(\frac{\partial\left(\boldsymbol{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u\right)}{\partial\boldsymbol{w}'}\right)_0\frac{\partial}{\partial s}\right]\boldsymbol{A}_0\boldsymbol{w}_m+\text{h.o.t.}
\end{aligned}\qquad [1.97]$$

Similarly, we obtain:

$$\begin{aligned}
&\boldsymbol{\mathcal{D}}_{uH}^{\star}\mathring{\boldsymbol{\sigma}}_u=\boldsymbol{\mathcal{D}}_{0uH}^{\star}\mathring{\boldsymbol{\sigma}}_u+\left[\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_{uH}^{\star}\mathring{\boldsymbol{\sigma}}_u\right)}{\partial\boldsymbol{w}}\right)_0+\left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_{uH}^{\star}\mathring{\boldsymbol{\sigma}}_u\right)}{\partial\boldsymbol{w}_m'}\right)_0\frac{\partial}{\partial s}\right]\boldsymbol{A}_0\boldsymbol{w}_m+\text{h.o.t.}\\
&\mathring{\boldsymbol{p}}=\mathring{\boldsymbol{p}}_0+\left(\frac{\partial\mathring{\boldsymbol{p}}}{\partial\boldsymbol{w}}\right)_0\boldsymbol{A}\boldsymbol{w}_m+\text{h.o.t.},\quad \mathring{\boldsymbol{P}}_H(\boldsymbol{w})=\mathring{\boldsymbol{P}}_{0H}+\left(\frac{\partial\mathring{\boldsymbol{P}}_H}{\partial\boldsymbol{w}}\right)_0\boldsymbol{A}_0\boldsymbol{w}_m+\text{h.o.t.}
\end{aligned}\qquad [1.98]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Hence, the field equations are linearized as follows (compare them with equation [1.58]):

$$\boldsymbol{A}_0^\star \boldsymbol{L}_u \boldsymbol{A}_0 \boldsymbol{w}_m + \boldsymbol{A}_0^\star \boldsymbol{G}_u \boldsymbol{A}_0 \boldsymbol{w}_m = \boldsymbol{A}_0^\star \tilde{\boldsymbol{p}}_0 \qquad [1.99]$$

and the boundary conditions as (compare them with equation [1.59]):

$$\begin{aligned}
&\left[\boldsymbol{\mathcal{A}}_0^\star \boldsymbol{L}_u \boldsymbol{A}_0 + \boldsymbol{\mathcal{L}}_{um} \boldsymbol{A}_0 + \boldsymbol{\mathcal{A}}_0^\star \boldsymbol{G}_u \boldsymbol{A}_0 + \boldsymbol{\mathcal{G}}_{um} \boldsymbol{A}_0\right]_H \boldsymbol{w}_m = \left[\tilde{\boldsymbol{P}}_{0m} + \boldsymbol{\mathcal{A}}_0^\star \tilde{\boldsymbol{p}}_0\right]_H \\
&\left[\boldsymbol{\mathcal{L}}_{us} \boldsymbol{A}_0 + \boldsymbol{\mathcal{G}}_{us} \boldsymbol{A}_0\right]_H \boldsymbol{w}_m = \left[\tilde{\boldsymbol{P}}_{0s}\right]_H \qquad [1.100] \\
&\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad \left[\boldsymbol{A}_{0s} \boldsymbol{w}_m\right]_H = \breve{\boldsymbol{w}}_{sH}
\end{aligned}$$

where the condensed elastic stiffness operators $\boldsymbol{L}_u, \boldsymbol{\mathcal{L}}_{umH}, \boldsymbol{\mathcal{L}}_{usH}$ have already been defined (equations [1.60]), and the condensed geometric stiffness operators are:

$$\begin{aligned}
\boldsymbol{G}_u &:= \left(\frac{\partial\left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial\left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s} - \left(\frac{\partial \mathring{\boldsymbol{p}}}{\partial \boldsymbol{w}}\right)_0 \\
\boldsymbol{\mathcal{G}}_{uH} &:= \left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_{uH}^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}}\right)_0 + \left(\frac{\partial\left(\boldsymbol{\mathcal{D}}_{uH}^\star \mathring{\boldsymbol{\sigma}}_u\right)}{\partial \boldsymbol{w}'}\right)_0 \frac{\partial}{\partial s} - \left(\frac{\partial \mathring{\boldsymbol{P}}_H}{\partial \boldsymbol{w}}\right)_0 + \dots
\end{aligned} \qquad [1.101]$$

with $\boldsymbol{\mathcal{G}}_{uH} := (\boldsymbol{\mathcal{G}}_{um}, \boldsymbol{\mathcal{G}}_{us})_H^T$. The latter are therefore the u-part of the operators $\boldsymbol{G}$, $\boldsymbol{\mathcal{G}}_H$ of the unconstrained beam, defined in equations [1.73], and also appearing in the mixed formulation for the constrained beam.

1.6 The variational formulation

In the previous sections, we formulated the Fundamental Problem of beam mechanics, via the *power balance approach*, based on the VPP, which provided the field equations and the alternative boundary conditions. We also mentioned the possibility of achieving the same goal by the *force balance approach* (when the beam is locally rigid), based on the application of the linear and angular momentum principles. There exists, however, a third method, which is called the *variational approach*, which we want to discuss here with a little detail.

A variational principle states that the solution to a given field problem renders stationary a (properly built-up) functional in its domain, i.e. in the space of functions from which the functional depends. The stationary condition, provided by the variational calculus, is a *differential* equation, which is called the *Eulerian equation* of the variational problem. In elastostatics, when the Fundamental Problem is formulated in the context of the displacement method, the proper functional is the *total potential energy* (TPE), which is a scalar function of the *admissible* vector

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

displacement field (i.e. compatible with the external and, possibly, internal constraints). With each arbitrarily chosen admissible displacement field, a scalar value of TPE is associated; by the stationary condition, we look for the particular vector field (possibly not unique) which makes the TPE “locally flat” in its neighborhood. This means that a first-order perturbation of the field that solves the Eulerian equations produces a second- or high-order perturbation in the value assumed by the TPE. The Eulerian equations supplied by the TPE principle are the balance equations and the mechanical boundary conditions we derived in alternate procedures, but, differently from those, *directly expressed in terms of displacements*.

The TPE principle, however, being related to an energy, only works for conservative systems[33]. When the beam is elastic (and therefore it cannot dissipate energy), we just have to assume that the external loads are conservative. However, if the request of conservativeness strongly limits the applicability of the variational approach, another circumstance mitigates this drawback, namely: *the first variation of the TPE is found to coincide with the VWP expression*, with stresses expressed in terms of displacements. Differently from the TPE principle, the VPP holds for any system, conservative and not, so that the varied form of TPE can be used as a method to automatically derive the VWP, to be applied, for example, to a non-conservative case. In this form, the Variational principle is said to be *extended*.

In this section, we will go over all the problems we studied in this chapter, by reobtaining known results via the variational approach.

1.6.1 The total potential energy principle

We define the TPE functional, $\mathit{\Pi}\left[\boldsymbol{w}\right]$, whose domain $\mathscr{U}$ is the space of the kinematically admissible displacements $\boldsymbol{w}$, as:

$$\mathit{\Pi}\left[\boldsymbol{w}\right] := U\left[\boldsymbol{w}\right] - W\left[\boldsymbol{w}\right] \qquad [1.102]$$

where $U\left[\boldsymbol{w}\right]$ is the elastic potential energy and $-W\left[\boldsymbol{w}\right]$ is the force potential energy (equal to the external work $W\left[\boldsymbol{w}\right]$, changed in sign, spent from the forces to bring the beam from the reference to the current configuration). Here:

$$\begin{aligned} U\left[\boldsymbol{w}\right] &:= \int_{\mathcal{S}} \phi(\boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right))ds \\ W\left[\boldsymbol{w}\right] &:= -\int_{\mathcal{S}} \psi(\boldsymbol{w})ds - \sum_{H=A}^{B} \mathit{\Psi}(\boldsymbol{w}_H) \end{aligned} \qquad [1.103]$$

33. Follower forces, therefore, are excluded.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where $\phi(\boldsymbol{\varepsilon})$ is the density of the elastic potential energy of the beam, $\psi(\boldsymbol{w}), \Psi(\boldsymbol{w}_H)$ are the potential energies of the forces and $\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right)$ expresses admissibility of strains and displacements (equation [1.1]).

The total potential energy principle (TPEP) states that the displacement field $\boldsymbol{w}$ that solves the elastic problem makes $\Pi\left[\boldsymbol{w}\right]$ stationary, i.e.:

$$\delta\Pi\left[\boldsymbol{w}\right] = 0 \qquad \forall\delta\boldsymbol{w} \in \mathscr{U} \tag{1.104}$$

Equivalently, we can say that *among all the kinematically admissible displacement fields, the ones also equilibrated render stationary the TPE*. By using the variational calculus, we find:

$$\begin{aligned}\delta\Pi\left[\boldsymbol{w}\right] &:= \int_{\mathcal{S}} \left(\frac{\partial\phi}{\partial\boldsymbol{\varepsilon}}\right)^T \delta\boldsymbol{\varepsilon} ds + \int_{\mathcal{S}} \left(\frac{\partial\psi}{\partial\boldsymbol{w}}\right)^T \delta\boldsymbol{w} ds + \sum_{H=A}^{B} \left(\frac{\partial\Psi}{\partial\boldsymbol{w}_H}\right)^T \delta\boldsymbol{w}_H \\ &= \int_{\mathcal{S}} \boldsymbol{\sigma}^T \delta\boldsymbol{\varepsilon} ds - \int_{\mathcal{S}} \boldsymbol{p}^T \delta\boldsymbol{w} ds - \sum_{H=A}^{B} \boldsymbol{P}_H^T \delta\boldsymbol{w}_H = 0 \quad \forall\left(\delta\boldsymbol{w}, \delta\boldsymbol{\varepsilon}\right) | \delta\boldsymbol{\varepsilon} = \boldsymbol{D}\delta\boldsymbol{w}\end{aligned} \tag{1.105}$$

where we accounted for the elastic law [1.14], the definition of force potential energies, $\boldsymbol{p} := -\partial\psi(\boldsymbol{w})/\partial\boldsymbol{w}$, $\boldsymbol{P}_H := -\partial\Psi(\boldsymbol{w}_H)/\partial\boldsymbol{w}_H$[34], and, finally, for the kinematic constraint [1.5] linking variations of strains and displacements. We, however, observe that equation [1.105] coincides with the VWP, equation [1.13], in which the stresses are expressed in terms of strains via the elastic law, and these, in turn, in terms of displacements, via the strain–displacement relationships. Therefore, *the TPE principle and the VWP are equivalent for conservative systems* and lead to the same balance equations (and boundary conditions). If, in contrast, forces are not conservative, then the extended form of the principle (i.e. the last line of equation [1.105]) also holds.

Dynamical systems: the Hamilton principle

When inertia forces have to be taken into account, we can either apply the d'Alembert principle, by including inertial effects in the external forces, or use the *Hamilton principle*. When specialized to the problem at hand, the principle states that *the true evolution* $\boldsymbol{w}_\star\left(s,t\right)$ *of a beam makes stationary the functional*:

$$\mathcal{H}\left[\boldsymbol{w}\right] := \int_{t_1}^{t_2} \left(T\left[\boldsymbol{w}\right] - \Pi\left[\boldsymbol{w}\right]\right) dt \tag{1.106}$$

34. The minus sign denotes a decrement of energy when the force spends a positive work.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

in the space of all the kinematically admissible motions $\boldsymbol{w}(s,t)$ *which bring the beam from a specified state* $\boldsymbol{w}(s,t_1)$ *to a specified state* $\boldsymbol{w}(s,t_2)$*, where* t_1, t_2 *are two selected times.* Here, T is the *kinetic energy* of the beam, and $\Pi := U - W$ is the TPE, already introduced. The Variational principle therefore requires that:

$$\delta\mathcal{H} := \delta \int_{t_1}^{t_2} (T - \Pi)\, dt = 0 \qquad \forall \delta\boldsymbol{w} | \delta\boldsymbol{w}(s,t_1) = \delta\boldsymbol{w}(s,t_2) = \mathbf{0} \qquad [1.107]$$

Its varied form:

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W)\, dt = 0 \qquad \forall \delta\boldsymbol{w} | \delta\boldsymbol{w}(s,t_1) = \delta\boldsymbol{w}(s,t_2) = \mathbf{0} \qquad [1.108]$$

is called the *extended hamilton principle*; it holds even for non-conservative forces (e.g. for visco-elastic or externally damped beams). When kinetic effects are negligible, the Hamilton principle reduces to the TPE principle.

1.6.2 Unconstrained beams

If we limit ourselves to linear elastic material, we have $\phi(\boldsymbol{\varepsilon}) = 1/2\boldsymbol{\mathcal{E}}^T\boldsymbol{E}\boldsymbol{\mathcal{E}}$ (equation [1.15]); moreover, if the external forces are dead loads $\boldsymbol{p}$, $\boldsymbol{P}_H$ (i.e. independent of $\boldsymbol{w}$), then, to within an inessential constant, is $\psi(\boldsymbol{w}) := -\boldsymbol{p}\boldsymbol{w}$, $\Psi(w_H) := -\boldsymbol{P}_H\boldsymbol{w}_H$, so that:

$$\Pi[\boldsymbol{w}] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}^T(\boldsymbol{w},\boldsymbol{w}')\,\boldsymbol{E}\boldsymbol{\mathcal{E}}(\boldsymbol{w},\boldsymbol{w}')\, ds - \int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}\, ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T\boldsymbol{P}_H \qquad [1.109]$$

By equating to zero the first variation, and observing that, for the symmetry of $\boldsymbol{E}$, it is $\delta\phi = \boldsymbol{\mathcal{E}}^T\boldsymbol{E}\delta\boldsymbol{\mathcal{E}} = (\boldsymbol{E}\boldsymbol{\mathcal{E}})^T\delta\boldsymbol{\mathcal{E}}$, where $\delta\boldsymbol{\mathcal{E}} = \boldsymbol{D}\delta\boldsymbol{w}$, equation [1.5], we have:

$$\begin{aligned}\delta\Pi[\boldsymbol{w}] &= \int_{\mathcal{S}} (\boldsymbol{E}\boldsymbol{\mathcal{E}})^T\boldsymbol{D}\delta\boldsymbol{w}\, ds - \int_{\mathcal{S}} \delta\boldsymbol{w}^T\boldsymbol{p}\, ds - \sum_{H=A}^{B}\delta\boldsymbol{w}_H^T\boldsymbol{P}_H \\ &= \int_{\mathcal{S}} \delta\boldsymbol{w}^T(\boldsymbol{D}^\star\boldsymbol{E}\boldsymbol{\mathcal{E}} - \boldsymbol{p})\, ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T(\boldsymbol{\mathcal{D}}^\star\boldsymbol{E}\boldsymbol{\mathcal{E}} - \boldsymbol{P})\right]_H = 0 \qquad \forall\delta\boldsymbol{w}\end{aligned} \qquad [1.110]$$

where we used the extended Green identity [1.8]. From equation [1.110], the balance equations [1.18] follow.

If we linearize the strain–displacement relationship, by taking $\boldsymbol{\mathcal{E}} = \boldsymbol{D}_0\boldsymbol{w}$, then, after integration by parts, the equilibrium operators $\boldsymbol{D}_0^\star$ and $\boldsymbol{\mathcal{D}}_{0H}^\star$ appear, so that the balance equations [1.20] of the linear theory are recovered.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

1.6.3 Constrained beams

When internal constraints exist, of type $\boldsymbol{\varepsilon}_c = \mathbf{0}$, the displacements $\boldsymbol{w}$ are no longer free, but they have to satisfy auxiliary equations $\boldsymbol{\mathcal{E}}_c(\boldsymbol{w}, \boldsymbol{w}') = \mathbf{0}$, which restrict the space $\mathscr{U}$ of the kinematically admissible displacements. To account for constraints, we can follow two different strategies, already discussed with reference to the VPP approach, i.e.: (a) to use Lagrange multipliers, according to the *mixed formulation*; and (b) to refer to master variables, identically satisfying the constraints, according to the *displacement formulation*. We briefly illustrate both the approaches.

The mixed formulation

By following the Lagrange multiplier technique, we modify the TPE functional (equation [1.109]) by adding to it a zero-quantity, namely the auxiliary conditions multiplied by unknown functions $\boldsymbol{\lambda} = \boldsymbol{\lambda}(s)$[35]; the modified TPE, therefore, reads:

$$\check{\Pi}[\boldsymbol{w}, \boldsymbol{\lambda}] := \Pi_u[\boldsymbol{w}] + \Pi_\lambda[\boldsymbol{w}, \boldsymbol{\lambda}] \qquad [1.111]$$

where $\Pi_u[\boldsymbol{w}]$ is the TPE of the unconstrained beam when $\boldsymbol{\varepsilon}_c = \mathbf{0}$, and $\Pi_\lambda[\boldsymbol{w}, \boldsymbol{\lambda}]$ the "work of the Lagrange multipliers in the zero-strains", namely:

$$\begin{aligned} \Pi_u[\boldsymbol{w}] &:= \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^T(\boldsymbol{w}, \boldsymbol{w}')\, \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u(\boldsymbol{w}, \boldsymbol{w}')\, ds - \int_{\mathcal{S}} \boldsymbol{w}^T \boldsymbol{p}\, ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T \boldsymbol{P}_H \\ \Pi_\lambda[\boldsymbol{w}, \boldsymbol{\lambda}] &:= \int_{\mathcal{S}} \boldsymbol{\lambda}^T \boldsymbol{\mathcal{E}}_c(\boldsymbol{w}, \boldsymbol{w}')\, ds \end{aligned} \qquad [1.112]$$

The variation of the first contribution, by remembering equation [1.110], is:

$$\delta\Pi_u[\boldsymbol{w}] = \int_{\mathcal{S}} \delta\boldsymbol{w}^T \left(\boldsymbol{D}_u^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \boldsymbol{p}\right) ds + \sum_{H=A}^{B} \left[\delta\boldsymbol{w}^T \left(\boldsymbol{\mathcal{D}}^\star{}_u \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \boldsymbol{P}\right)\right]_H \qquad [1.113]$$

The variation of the second contribution, since $\delta\left(\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c\right) = \delta\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c + \boldsymbol{\lambda}^T\delta\boldsymbol{\mathcal{E}}_c$, reads:

$$\begin{aligned} \delta\Pi_\lambda[\boldsymbol{w}, \boldsymbol{\lambda}] &= \int_{\mathcal{S}} \left(\delta\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c + \boldsymbol{\lambda}^T \boldsymbol{D}_c \delta\boldsymbol{w}\right) ds \\ &= \int_{\mathcal{S}} \left(\delta\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c + \delta\boldsymbol{w}^T \boldsymbol{D}_c^\star \boldsymbol{\lambda}\right) ds + \sum_{H=A}^{B} \left[\delta\boldsymbol{w}^T \boldsymbol{\mathcal{D}}_c^\star \boldsymbol{\lambda}\right]_H \end{aligned} \qquad [1.114]$$

35. The *constrained* problem "finds the function $u(s)$ which makes the functional $I[u(s)] := \int_a^b \mathcal{L}(u(s), u'(s))\, ds$ stationary, under the differential constraints $f_i(u(s), u'(s)) = 0,\ i = 1, \ldots, n$", is equivalent to the *unconstrained* problem: "it finds the functions $u(s)$ and $\lambda_i(s)$ that makes stationary the modified functional $\check{I}[u(s), \lambda_i(s)] := \int_a^b \mathcal{L}(u(s), u'(s))\, ds + \sum_{i=1}^n \int_a^b \lambda_i(s) f_i(u(s), u'(s))\, ds$".

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

having accounted for $\delta\boldsymbol{\mathcal{E}}_c = \boldsymbol{D}_c\delta\boldsymbol{w}$ and performed an integration by parts according to equation [1.8].

The Variational principle finally reads:

$$\begin{aligned}\delta\check{\Pi}\left[\boldsymbol{w},\boldsymbol{\lambda}\right] = & \int_{\mathcal{S}} \left[\delta\boldsymbol{w}^T\left(\boldsymbol{D}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{D}_c^\star\boldsymbol{\lambda} - \boldsymbol{p}\right) + \delta\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c\right] ds \\ & + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\left(\boldsymbol{\mathcal{D}}^\star{}_u\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{\mathcal{D}}_c^\star\boldsymbol{\lambda} - \boldsymbol{P}\right)\right]_H = 0 \qquad \forall\left(\delta\boldsymbol{w},\delta\lambda\right)\end{aligned} \quad [1.115]$$

from which the constrained elastic problem, equations [1.37] and [1.38], follows, with $\boldsymbol{\lambda} \equiv \boldsymbol{\sigma}_c$.

If we linearize the strain–displacement relationship, by taking $\boldsymbol{\mathcal{E}}_u = \boldsymbol{D}_{0u}\boldsymbol{w}$, $\boldsymbol{\mathcal{E}}_c = \boldsymbol{D}_{0c}\boldsymbol{w}$, then, after integration by parts, the equilibrium operators $\boldsymbol{D}_{0u}^\star$, $\boldsymbol{D}_{0c}^\star$ and $\boldsymbol{\mathcal{D}}_{0uH}^\star$, $\boldsymbol{\mathcal{D}}_{0cH}^\star$ appear, so that the balance equations [1.39] and [1.40] of the linear theory are recovered.

The displacement formulation

Instead of using Lagrange multipliers, we consider a TPE reduced to the unconstrained contribution $\Pi_u\left[\boldsymbol{w}\right]$ (equation [1.112a]), whose domain $\mathscr{U}_m := \left\{\boldsymbol{w}|\boldsymbol{w}\in\mathscr{U},\ \boldsymbol{w} = \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)\right\}$ is a *subset* of $\mathscr{U}$, where $\boldsymbol{w}_m$ are master variables identically satisfying the constraints, i.e. $\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{\mathcal{W}}, \frac{\partial}{\partial s}\boldsymbol{\mathcal{W}}\right) = \boldsymbol{0}$, $\forall\boldsymbol{w}_m$. Therefore, the TPE is sided by constraints as follows:

$$\begin{aligned}&\Pi_u\left[\boldsymbol{w}\right] := \frac{1}{2}\int_{\mathcal{S}}\boldsymbol{\mathcal{E}}_u^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds - \int_{\mathcal{S}}\boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H \\ &\boldsymbol{w} = \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)\end{aligned} \quad [1.116]$$

Constraints [1.116b] could be easily accounted for by direct substitution in equation [1.116a], by leading to a new functional $\Pi_u\left[\boldsymbol{w}_m\right] := \Pi_u\left[\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)\right]$ in which $\boldsymbol{w}_m$ are free variables. However, we find it more convenient first to perform the variation $\delta\Pi_u\left[\boldsymbol{w}\right]$ (already performed in equation [1.113]) and then to substitute the constraints, both in the arguments (e.g. $\boldsymbol{\mathcal{E}}_u = \boldsymbol{\mathcal{E}}_u\left(\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right), \frac{\partial}{\partial s}\boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)\right)$), and in the variation, i.e. $\delta\boldsymbol{w} = \boldsymbol{A}\delta\boldsymbol{w}_m$ (having used equation [1.43], multiplied by dt). In so doing, we obtain:

$$\begin{aligned}\delta\Pi_u\left[\boldsymbol{w}_m\right] = & \int_{\mathcal{S}}\left(\boldsymbol{D}_u^\star\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \boldsymbol{p}\right)^T\left(\boldsymbol{A}\delta\boldsymbol{w}_m\right)ds \\ & + \sum_{H=A}^{B}\left[\left(\boldsymbol{A}\delta\boldsymbol{w}_m\right)^T\left(\boldsymbol{\mathcal{D}}^\star{}_u\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \boldsymbol{P}\right)\right]_H = 0 \qquad \forall\delta\boldsymbol{w}_m\end{aligned} \quad [1.117]$$

However, this is just equation [1.50], with the active stresses expressed in terms of strains. Therefore, by performing similar steps, the Fundamental Problem equations [1.56], [1.57] are recovered.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

If we linearize kinematics, by taking $\boldsymbol{\mathcal{E}}_u = \boldsymbol{D}_{0u}\boldsymbol{w}$, $\boldsymbol{A} = \boldsymbol{A}_0 = (\boldsymbol{I}_m, \boldsymbol{A}_{0s})^T$, then, after integration by parts, the operators $\boldsymbol{D}^\star_{0u}, \boldsymbol{A}_0$ and $\boldsymbol{\mathcal{D}}^\star_{0H}, \boldsymbol{\mathcal{A}}^\star_{0H}$ appear, so that the balance equations [1.59] of the linear theory are recovered.

1.6.4 Unconstrained prestressed beams

When preloads and prestresses act on the beam, the TPE [1.102] must accordingly be modified. By remembering expression [1.66] of the elastic potential and considering total dead loads, we have:

$$\mathit{\Pi}\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] := \mathit{\Pi}\left[\boldsymbol{w}\right] + \mathring{\mathit{\Pi}}\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] \qquad [1.118]$$

where $\mathit{\Pi}\left[\boldsymbol{w}\right]$ is the TPE of the unprestressed beam (equation [1.109], with incremental loads $\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{P}}_H$ replacing $\boldsymbol{p}, \boldsymbol{P}_H$), and:

$$\mathring{\mathit{\Pi}}\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] := \int_{\mathcal{S}} \mathring{\boldsymbol{\sigma}}^T \boldsymbol{\mathcal{E}}\left(\boldsymbol{w}, \boldsymbol{w}'\right) ds - \int_{\mathcal{S}} \boldsymbol{w}^T \mathring{\boldsymbol{p}} ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T \mathring{\boldsymbol{P}}_H \qquad [1.119]$$

is the contribution of prestresses and preloads. $\delta\left(\mathring{\boldsymbol{\sigma}}^T \boldsymbol{\mathcal{E}}\right) = \mathring{\boldsymbol{\sigma}}^T \boldsymbol{D}\delta\boldsymbol{w}$, we have, after integration by parts:

$$\delta\mathring{\mathit{\Pi}}\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] := \int_{\mathcal{S}} \delta\boldsymbol{w}^T \left(\boldsymbol{D}^\star \mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{p}}\right) ds + \sum_{H=A}^{B} \left[\delta\boldsymbol{w}^T \left(\boldsymbol{\mathcal{D}}^\star \mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{P}}\right)\right]_H \qquad [1.120]$$

The Variational principle $\delta\mathit{\Pi}\left[\boldsymbol{w}\right] + \delta\mathring{\mathit{\Pi}}\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] = 0$ then leads to the balance equations [1.69], where the elastic and incremental load terms spring from the first contribution (see equation [1.110]), and prestress and preload terms stem from the second.

The linearized theory

The variational formulation is often followed in literature in the context of the linearized theory of prestressed beams (under conservative loads). The main idea of the method consists of assuming a *quadratic* polynomial expression for the TPE, in order to get equilibrium equations *linear* in the displacements (given that the variation entails a lowering of 1 in the polynomial degree). Therefore, we write the strains by series expansions, as the sum of linear and quadratic contributions in the displacements, namely:

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(1)}\left(\boldsymbol{w}, \boldsymbol{w}'\right) + \boldsymbol{\mathcal{E}}^{(2)}\left(\boldsymbol{w}, \boldsymbol{w}'\right) + \text{h.o.t.} \qquad [1.121]$$

and, moreover, we assume the preloads as $\mathrm{O}(1)$-quantities and the incremental load as $\mathrm{O}(\boldsymbol{w})$-quantities. Hence, the TPE [1.118] reads as:

$$\mathit{\Pi}\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] = \mathit{\Pi}^{(1)}\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] + \mathit{\Pi}^{(2)}\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] + \text{h.o.t.} \qquad [1.122]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where, by omitting the arguments:

$$\mathit{\Pi}^{(1)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right]:=\int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}^{T}\boldsymbol{\mathcal{E}}^{(1)}ds-\int_{\mathcal{S}}\boldsymbol{w}^{T}\mathring{\boldsymbol{p}}ds-\sum_{H=A}^{B}\boldsymbol{w}_{H}^{T}\mathring{\boldsymbol{P}}_{H}$$
$$\mathit{\Pi}^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right]:=\int_{\mathcal{S}}\left(\frac{1}{2}\boldsymbol{\mathcal{E}}^{(1)T}\boldsymbol{E}\boldsymbol{\mathcal{E}}^{(1)}+\mathring{\boldsymbol{\sigma}}^{T}\boldsymbol{\mathcal{E}}^{(2)}\right)ds-\int_{\mathcal{S}}\boldsymbol{w}^{T}\tilde{\boldsymbol{p}}ds-\sum_{H=A}^{B}\boldsymbol{w}_{H}^{T}\tilde{\boldsymbol{P}}_{H} \qquad [1.123]$$

are first- and second-order terms, respectively. However, we observe that $\delta\mathit{\Pi}^{(1)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right]=0\ \forall\delta\boldsymbol{w}$, since it expresses the total virtual work spent by the equilibrated prestresses and preloads, acting in the reference configuration, in the kinematically admissible infinitesimal strains $\boldsymbol{\mathcal{E}}^{(1)}$ and infinitely small displacements $\boldsymbol{w}$. Hence, the first-order term of the potential energy is not essential, and we can assume, after truncation, $\mathit{\Pi}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right]\equiv\mathit{\Pi}^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right]$.

The first- and second-order parts of the strain components read:

$$\mathcal{E}_i^{(1)}:=\left(\frac{\partial\mathcal{E}_i}{\partial\boldsymbol{w}}\right)_0^T\boldsymbol{w}+\left(\frac{\partial\mathcal{E}_i}{\partial\boldsymbol{w}'}\right)_0^T\boldsymbol{w}'$$
$$\mathcal{E}_i^{(2)}:=\frac{1}{2}\left(\boldsymbol{w}^T\boldsymbol{A}_i\boldsymbol{w}+2\boldsymbol{w}^T\boldsymbol{B}_i\boldsymbol{w}'+\boldsymbol{w}'^T\boldsymbol{C}_i\boldsymbol{w}'\right) \qquad [1.124]$$

where we used the positions [1.77]. When the variational principle $\delta\mathit{\Pi}^{(2)}=0,\ \forall\delta\boldsymbol{w}$ is invoked, the first, third and fourth addenda in equation [1.123b] lead, after straightforward (and therefore omitted) calculations, to the familiar terms of the linear theory of the stress-free beam. In contrast, we focus the attention on the second term, whose variation reads:

$$\begin{aligned}\delta\int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}^T\boldsymbol{\mathcal{E}}^{(2)}ds&=\sum_{i=1}^{M}\int_{\mathcal{S}}\mathring{\sigma}_i\delta\mathcal{E}_i^{(2)}ds\\&=\sum_{i=1}^{M}\int_{\mathcal{S}}\mathring{\sigma}_i\left(\delta\boldsymbol{w}^T\boldsymbol{A}_i\boldsymbol{w}+\delta\boldsymbol{w}^T\boldsymbol{B}_i\boldsymbol{w}'+\boldsymbol{w}^T\boldsymbol{B}_i\delta\boldsymbol{w}'+\delta\boldsymbol{w}'^T\boldsymbol{C}_i\boldsymbol{w}'\right)ds\\&=\sum_{i=1}^{M}\int_{\mathcal{S}}\delta\boldsymbol{w}^T\left[\left(\boldsymbol{A}_i\boldsymbol{w}+\boldsymbol{B}_i\boldsymbol{w}'\right)\mathring{\sigma}_i-\left(\boldsymbol{B}_i^T\boldsymbol{w}\mathring{\sigma}_i+\boldsymbol{C}_i\boldsymbol{w}'\mathring{\sigma}_i\right)'\right]ds+\\&+\left[\delta\boldsymbol{w}^T\left(\boldsymbol{B}_i^T\boldsymbol{w}\mathring{\sigma}_i+\boldsymbol{C}_i\boldsymbol{w}'\mathring{\sigma}_i\right)\right]_A^B\end{aligned} \qquad [1.125]$$

where we accounted for the symmetry of $\boldsymbol{A}_i,\boldsymbol{C}_i$ and integrated by parts. By remembering equations [1.78] and [1.81], we can write:

$$\delta\int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}^T\boldsymbol{\mathcal{E}}^{(2)}ds=\int_{\mathcal{S}}\delta\boldsymbol{w}^T\boldsymbol{G}\boldsymbol{w}ds+\sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\boldsymbol{\mathcal{G}}\boldsymbol{w}\right]_H \qquad [1.126]$$

to within, of course, the effects of the follower preloads, absent here. Therefore, the variational principle leads to balance equation [1.72].

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

REMARK 1.10. The two contributions under the integral sign in $\Pi^{(2)}$ represent, in order: (a) the elastic potential of a stress-free beam, when kinematics is linearized; and (b) *the work spent by the prestress in the second-order part of the strain–displacement relationship*. While the first term behaves as the progenitor of the linear elastic stiffnesses $\boldsymbol{L}$ and $\boldsymbol{\mathcal{L}}_H$, the second term is the progenitor of the geometric stiffnesses $\boldsymbol{G}$ and $\boldsymbol{\mathcal{G}}_H$.

1.6.5 Constrained prestressed beams

The nonlinear mixed formulation

We already introduced in equation [1.112] a modified TPE $\breve{\Pi}[\boldsymbol{w}, \boldsymbol{\lambda}]$ for unprestressed beams, able to account for the constraints $\boldsymbol{\mathcal{E}}_c(\boldsymbol{w}, \boldsymbol{w}') = \boldsymbol{0}$ via the Lagrange multipliers. Now, we just have to update the expression of the elastic potential to include the contribution of prestress, and to add the potential of the preloads, as we did in equation [1.118]. Therefore, we have:

$$\breve{\Pi}[\boldsymbol{w}, \boldsymbol{\lambda}; \mathring{\boldsymbol{\sigma}}_u] := \Pi_u[\boldsymbol{w}] + \Pi_\lambda[\boldsymbol{w}, \boldsymbol{\lambda}] + \mathring{\Pi}[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}_u] \qquad [1.127]$$

where $\Pi_u[\boldsymbol{w}]$ is the TPE of the unconstrained beam when $\boldsymbol{\varepsilon}_c = \boldsymbol{0}$ (equation [1.112a), $\Pi_\lambda[\boldsymbol{w}, \boldsymbol{\lambda}]$ is the work of the Lagrange multipliers on the constrained zero-strains (equation [1.112b]), and the additional term is[36]:

$$\mathring{\Pi}[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}_u] := \int_{\mathcal{S}} \mathring{\boldsymbol{\sigma}}_u^T \boldsymbol{\mathcal{E}}_u(\boldsymbol{w}, \boldsymbol{w}')\, ds - \int_{\mathcal{S}} \boldsymbol{w}^T \mathring{\boldsymbol{p}} ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T \mathring{\boldsymbol{P}}_H \qquad [1.128]$$

The variations of the first two contributions are given by equations [1.113] and [1.114]; the variation of the third contribution is:

$$\delta\mathring{\Pi}[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}] := \int_{\mathcal{S}} \delta\boldsymbol{w}^T \left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{p}}\right) ds + \sum_{H=A}^{B} \left[\delta\boldsymbol{w}^T \left(\boldsymbol{\mathcal{D}}^\star{}_u \mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}\right)\right]_H \qquad [1.129]$$

Hence, the variational principle reads:

$$\begin{aligned}\delta\breve{\Pi}[\boldsymbol{w}, \boldsymbol{\lambda}; \mathring{\boldsymbol{\sigma}}_u] = {} & \int_{\mathcal{S}} \delta\boldsymbol{w}^T \left(\boldsymbol{D}_u^\star \boldsymbol{E}_{uu} \boldsymbol{\mathcal{E}}_u - \tilde{\boldsymbol{p}}\right) ds + \sum_{H=A}^{B} \left[\delta\boldsymbol{w}^T \left(\boldsymbol{\mathcal{D}}_u^\star \boldsymbol{E}_{uu} \boldsymbol{\mathcal{E}}_u - \tilde{\boldsymbol{P}}\right)\right]_H \\ & + \int_{\mathcal{S}} \left[\delta\boldsymbol{w}^T \left(\boldsymbol{D}_u^\star \mathring{\boldsymbol{\sigma}}_u + \boldsymbol{D}_c^\star \boldsymbol{\lambda} - \mathring{\boldsymbol{p}}\right) + \delta\boldsymbol{\lambda}^T \boldsymbol{\mathcal{E}}_c\right] ds + \\ & + \sum_{H=A}^{B} \left[\delta\boldsymbol{w}^T \left(\boldsymbol{\mathcal{D}}_u^\star \mathring{\boldsymbol{\sigma}}_u + \boldsymbol{\mathcal{D}}_c^\star \boldsymbol{\lambda} - \mathring{\boldsymbol{P}}\right)\right]_H = 0 \quad \forall(\delta\boldsymbol{w}, \delta\boldsymbol{\lambda})\end{aligned}$$

[1.130]

From the latter, the boundary value problems [1.87] and [1.88] follow, if $\boldsymbol{\lambda} := \mathring{\boldsymbol{\sigma}}_c + \tilde{\boldsymbol{\sigma}}_c$ is taken.

36. This is analogous to that in equation [1.119], relevant to the unconstrained beam, but it is limited to the admissible strains.

The linearized mixed formulation

According to the linearized theory, we have to retain in equation [1.127] the second-order terms only. Since $\boldsymbol{\lambda} = \mathring{\boldsymbol{\sigma}}_c + \tilde{\boldsymbol{\sigma}}_c$ is a sum of a zero-th order and a first-order term, then $\boldsymbol{\lambda}^T \boldsymbol{\mathcal{E}}_c^{(2)} = \mathring{\boldsymbol{\sigma}}_c^T \boldsymbol{\mathcal{E}}_c^{(2)} + \tilde{\boldsymbol{\sigma}}_c^T \boldsymbol{\mathcal{E}}_c^{(1)} + \text{h.o.t.}$; therefore[37]:

$$\begin{aligned}\check{\Pi}^{(2)}\left[\boldsymbol{w}, \tilde{\boldsymbol{\sigma}}_c; \mathring{\boldsymbol{\sigma}}\right] := & \int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^{(1)T}(\boldsymbol{w}, \boldsymbol{w}') \boldsymbol{E}_{uu} \boldsymbol{\mathcal{E}}_u^{(1)}(\boldsymbol{w}, \boldsymbol{w}') ds \\ & - \int_{\mathcal{S}} \boldsymbol{w}^T \tilde{\boldsymbol{p}} ds - \sum_{H=A}^{B} \boldsymbol{w}_H^T \tilde{\boldsymbol{P}}_H + \int_{\mathcal{S}} \left(\mathring{\boldsymbol{\sigma}}^T \boldsymbol{\mathcal{E}}^{(2)} + \tilde{\boldsymbol{\sigma}}_c \boldsymbol{D}_{0c} \boldsymbol{w}\right) ds\end{aligned} \qquad [1.131]$$

where we accounted for $\boldsymbol{\mathcal{E}}_c^{(1)} = \boldsymbol{D}_{0c} \boldsymbol{w}$ and we merged two terms.

The variational principle, by remembering equations [1.126], therefore reads:

$$\begin{aligned}\delta \check{\Pi}^{(2)}\left[\boldsymbol{w}, \tilde{\boldsymbol{\sigma}}_c; \mathring{\boldsymbol{\sigma}}\right] = & \int_{\mathcal{S}} \delta \boldsymbol{w}^T \left(\boldsymbol{L}_u \boldsymbol{w} - \tilde{\boldsymbol{p}}\right) ds + \sum_{H=A}^{B} \left[\delta \boldsymbol{w}^T \left(\boldsymbol{\mathcal{L}}_u \boldsymbol{w} - \tilde{\boldsymbol{P}}\right)\right]_H \\ & + \int_{\mathcal{S}} \delta \boldsymbol{w}^T \boldsymbol{G} \boldsymbol{w} ds + \sum_{H=A}^{B} \left[\delta \boldsymbol{w}^T \boldsymbol{\mathcal{G}} \boldsymbol{w}\right]_H \\ & + \int_{\mathcal{S}} \left(\delta \tilde{\boldsymbol{\sigma}}_c^T \boldsymbol{\mathcal{E}}_c + \delta \boldsymbol{w}^T \boldsymbol{D}_{0c}^{\star} \tilde{\boldsymbol{\sigma}}_c\right) ds + \sum_{H=A}^{B} \delta \boldsymbol{w}_H^T \boldsymbol{\mathcal{D}}_{0cH}^{\star} \tilde{\boldsymbol{\sigma}}_c \quad \forall \left(\delta \boldsymbol{w}, \delta \tilde{\boldsymbol{\sigma}}_c\right)\end{aligned} \qquad [1.132]$$

from which equations [1.89] and [1.90] are recovered, together with the constraint equation.

The nonlinear displacement formulation

We write the TPE $\Pi_u[\boldsymbol{w}]$ with the geometrical constraint appended, as we did for the unprestressed beam (equation [1.116]), but we add the prestress contribution $\mathring{\Pi}[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}]$ (equation [1.128]):

$$\begin{aligned}\Pi\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] := & \Pi_u[\boldsymbol{w}] + \mathring{\Pi}\left[\boldsymbol{w}; \mathring{\boldsymbol{\sigma}}\right] \\ \boldsymbol{w} = & \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right)\end{aligned} \qquad [1.133]$$

37. Note that the free variables are now $\boldsymbol{w}, \tilde{\boldsymbol{\sigma}}_c$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By following the same steps of the unprestressed case, we perform the variation, substitute $\delta\boldsymbol{\mathcal{E}}_u = \boldsymbol{D}_u\delta\boldsymbol{w}$ and integrate by parts, to obtain:

$$\begin{aligned}\delta\Pi\left[\mathbf{w};\mathring{\boldsymbol{\sigma}}\right] &= \int_{\mathcal{S}} \delta\mathbf{w}^T\left(\mathbf{D}_u^{\star}\mathbf{E}_{uu}\boldsymbol{\mathcal{E}}_u - \tilde{\mathbf{p}}\right)ds + \sum_{H=A}^{B}\left[\delta\mathbf{w}^T\left(\boldsymbol{\mathcal{D}}_u^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \tilde{\mathbf{P}}\right)\right]_H \\ &+ \int_{\mathcal{S}} \delta\mathbf{w}^T\left(\mathbf{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u - \mathring{\mathbf{p}}\right)ds + \sum_{H=A}^{B}\left[\delta\mathbf{w}^T\left(\boldsymbol{\mathcal{D}}_u^{\star}\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}\right)\right]_H = 0 \qquad \forall\delta\boldsymbol{w} = \boldsymbol{A}\delta\boldsymbol{w}_m\end{aligned}$$

[1.134]

Then, we substitute the constraint, both in the arguments and in the variation (i.e. $\delta\boldsymbol{w} = \boldsymbol{A}\left(\boldsymbol{w}_m, \boldsymbol{w}_m'\right)\delta\boldsymbol{w}_m$), thus obtaining:

$$\begin{aligned}\delta\Pi\left[\boldsymbol{w}_m;\mathring{\boldsymbol{\sigma}}\right] &= \int_{\mathcal{S}} \left(\left(\boldsymbol{D}_u^{\star}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u - \tilde{\boldsymbol{p}}\right) + \left(\boldsymbol{D}_u^{\star}\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{p}}\right)\right)^T\left(\boldsymbol{A}\delta\boldsymbol{w}_m\right)ds \\ &+ \sum_{H=A}^{B}\left[\left(\boldsymbol{A}\delta\boldsymbol{w}_m\right)^T\left(\boldsymbol{\mathcal{D}^{\star}}_u\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \left(\boldsymbol{\mathcal{D}^{\star}}_u\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}\right)\right)\right]_H = 0 \qquad \forall\delta\boldsymbol{w}_m\end{aligned}$$

[1.135]

By integrating by parts with the aid of equation [1.51], equation [1.95] is recovered, with the boundary conditions [1.96].

The linearized displacement formulation

When the TPE [1.133a] is truncated at the second order, and the constraint [1.133b] is linearized, they become:

$$\begin{aligned}\Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}_u\right] &:= \int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^{(1)T}(\boldsymbol{w},\boldsymbol{w}')\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u(\boldsymbol{w},\boldsymbol{w}')ds \\ &- \int_{\mathcal{S}} \boldsymbol{w}^T\tilde{\boldsymbol{p}}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\tilde{\boldsymbol{P}}_H + \int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}_u^T\boldsymbol{\mathcal{E}}_u^{(2)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds \\ \boldsymbol{w} &= \boldsymbol{A}_0\boldsymbol{w}_m\end{aligned} \qquad [1.136]$$

The variational principle reads:

$$\begin{aligned}\delta\Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}_u\right] &= \int_{\mathcal{S}}\delta\boldsymbol{w}^T\left(\boldsymbol{L}_u\boldsymbol{w} - \tilde{\boldsymbol{p}}\right)ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\left(\boldsymbol{\mathcal{L}}_u\boldsymbol{w} - \tilde{\boldsymbol{P}}\right)\right]_H \\ &+ \int_{\mathcal{S}}\delta\boldsymbol{w}^T\boldsymbol{G}_u\boldsymbol{w}ds + \sum_{H=A}^{B}\left[\delta\boldsymbol{w}^T\boldsymbol{\mathcal{G}}_u\boldsymbol{w}\right]_H = 0 \qquad \forall\delta\boldsymbol{w} = \boldsymbol{A}_0\delta\boldsymbol{w}_m\end{aligned} \qquad [1.137]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where, concerning the geometric term, we used equation [1.126] and exploited similarity between the definitions [1.78], [1.81] for $\boldsymbol{G}$, $\boldsymbol{\mathcal{G}}_H$, and definitions [1.101] for $\boldsymbol{G}_u$, $\boldsymbol{\mathcal{G}}_{uH}$.

By substituting the constraint, namely $\boldsymbol{w} = \boldsymbol{A}_0\boldsymbol{w}_m,\ \delta\boldsymbol{w} = \boldsymbol{A}_0\delta\boldsymbol{w}_m$, we obtain:

$$\int_{\mathcal{S}} \left(\boldsymbol{A}_0\delta\boldsymbol{w}_m\right)^T \left(\left(\boldsymbol{L}_u + \boldsymbol{G}_u\right)\left(\boldsymbol{A}_0\boldsymbol{w}_m\right) - \tilde{\boldsymbol{p}}\right) ds + \sum_{H=A}^{B} \left[\left(\boldsymbol{A}_0\delta\boldsymbol{w}_m\right)^T \left(\left(\boldsymbol{\mathcal{L}}_u + \boldsymbol{\mathcal{G}}_u\right)\left(\boldsymbol{A}_0\boldsymbol{w}_m\right) - \tilde{\boldsymbol{P}}\right)\right]_H = 0 \qquad \forall\delta\boldsymbol{w}_m \tag{1.138}$$

Then, by integrating by parts (with the help of the linearized version of equation [1.51], i.e. for $\boldsymbol{A}$ replaced by $\boldsymbol{A}_0$), and by splitting the boundary terms into master and slave contributions, i.e. by letting $\boldsymbol{A}_0 = \left(\boldsymbol{I}_m, \boldsymbol{A}_{0s}\right)^T$, we get:

$$\begin{aligned}
&\int_{\mathcal{S}} \delta\boldsymbol{w}_m^T \boldsymbol{A}_0^\star \left(\left(\boldsymbol{L}_u + \boldsymbol{G}_u\right)\left(\boldsymbol{A}_0\boldsymbol{w}_m\right) - \tilde{\boldsymbol{p}}\right) ds \\
&+ \sum_{H=A}^{B} \left[\delta\boldsymbol{w}_m^T \boldsymbol{\mathcal{A}}_0^\star \left(\left(\boldsymbol{L}_u + \boldsymbol{G}_u\right)\left(\boldsymbol{A}_0\boldsymbol{w}_m\right) - \tilde{\boldsymbol{p}}\right)\right]_H \\
&+ \sum_{H=A}^{B} \left[\delta\boldsymbol{w}_m^T \left(\left(\boldsymbol{\mathcal{L}}_{um} + \boldsymbol{\mathcal{G}}_{um}\right)\left(\boldsymbol{A}_0\boldsymbol{w}_m\right) - \tilde{\boldsymbol{P}}_m\right)\right]_H \\
&+ \sum_{H=A}^{B} \left[\left(\boldsymbol{A}_{0s}\delta\boldsymbol{w}_m\right)^T \left(\left(\boldsymbol{\mathcal{L}}_{us} + \boldsymbol{\mathcal{G}}_{us}\right)\left(\boldsymbol{A}_0\boldsymbol{w}_m\right) - \tilde{\boldsymbol{P}}_s\right)\right]_H = 0 \qquad \forall\delta\boldsymbol{w}_m
\end{aligned} \tag{1.139}$$

From this form, the field equations [1.99] and the boundary conditions [1.100] follow.

1.7 Example: the linear Timoshenko beam

Let us consider the unconstrained linear model of the Timoshenko beam, undergoing transverse displacements and rotations only, governed by the following equations:

$$\begin{aligned}
&\begin{pmatrix} \gamma \\ \kappa \end{pmatrix} = \begin{pmatrix} u' - \theta \\ \theta' \end{pmatrix}, \quad \begin{pmatrix} -\partial_s & 0 \\ -1 & -\partial_s \end{pmatrix} \begin{pmatrix} T \\ M \end{pmatrix} = \begin{pmatrix} p \\ c \end{pmatrix}, \\
&\begin{pmatrix} T \\ M \end{pmatrix} = \begin{pmatrix} GA_t & 0 \\ 0 & EJ \end{pmatrix} \begin{pmatrix} \gamma \\ \kappa \end{pmatrix}
\end{aligned} \tag{1.140}$$

with the boundary conditions (a geometrical one excludes the dual mechanical):

$$\begin{pmatrix} u_H \\ \theta_H \end{pmatrix} = \begin{pmatrix} \breve{u}_H \\ \breve{\theta}_H \end{pmatrix}, \qquad \begin{pmatrix} \mp 1 & 0 \\ 0 & \mp 1 \end{pmatrix} \begin{pmatrix} T_H \\ M_H \end{pmatrix} = \begin{pmatrix} P_H \\ C_H \end{pmatrix}, \qquad H = A, B \tag{1.141}$$

The previous equations are, in order: the (infinitesimal) strain–displacement relationships; the equilibrium equations (in the reference configuration); and the elastic law. Here,

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$\boldsymbol{\varepsilon} = (\gamma, \kappa)^T$ are generalized strains, shear–strain and curvature, respectively; $\boldsymbol{w} = (u, \theta)^T$ are generalized displacements, transverse displacement and rotation, respectively; $\boldsymbol{\sigma} = (T, M)^T$ are generalized stresses, shear–force and bending moment, respectively; $\boldsymbol{p} = (p, c)^T$ are external forces, transverse and couples; and GA_t, EJ are elastic stiffnesses. Moreover, $\breve{\boldsymbol{w}}_H = \left(\breve{u}_H, \breve{\theta}_H\right)^T$ are prescribed displacements/rotations and $\boldsymbol{P}_H = (P_H, C_H)^T$ are prescribed forces/couples at ends. The minus/plus identity matrices are the boundary equilibrium operator $\boldsymbol{\mathcal{D}}^\star_H$. The strain-rate-velocity relationships read:

$$\begin{pmatrix} \dot{\gamma} \\ \dot{\kappa} \end{pmatrix} = \begin{pmatrix} \partial_s & -1 \\ 0 & \partial_s \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{\theta} \end{pmatrix} \qquad [1.142]$$

which defines the operator $\boldsymbol{D} \equiv \boldsymbol{D}_0$, adjoint of the equilibrium operator $\boldsymbol{D}^\star \equiv \boldsymbol{D}^\star_0$ appearing in equation [1.140b].

For this model, we want to enforce the constraint condition $\gamma = 0$ (unshearable beam) and derive the displacement formulation (Euler–Bernoulli beam). Then, we want to find the reactive stress T.

The admissible strain is κ, the constrained strain is γ; consistently, T is the reactive stress and M is the active stress. In the constraint equation $u' - \theta = 0$, we chose the non-differentiated variable θ as slave variable and, consequently, u as master variable. Accordingly, $\boldsymbol{D}_u = (0, \partial_s)$, $\boldsymbol{D}_c = (\partial_s, -1)$ and $\boldsymbol{D}^\star_u = (0, -\partial_s)^T$, $\boldsymbol{D}^\star_c = (-\partial_s, -1)^T$. From the constraint, we get $\theta = u'$, and therefore $\boldsymbol{\mathcal{W}} = (u, u')^T$ (equation [1.42]); by time-differentiating it, we obtain (equation [1.43]):

$$\begin{pmatrix} \dot{u} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 1 \\ \partial_s \end{pmatrix} \dot{u} \qquad [1.143]$$

which defines the velocity constraint operator $\boldsymbol{A} := (1, \partial_s)^T$. Note that $\boldsymbol{D}_c\boldsymbol{A} = \partial_s - \partial_s = 0$.

The condensed strain–displacement relationships [1.45] and the condensed strain-rate-velocities [1.47], read:

$$\kappa = \left(u'\right)', \qquad \dot{\kappa} = (0, \partial_s) \begin{pmatrix} 1 \\ \partial_s \end{pmatrix} \dot{u} = \dot{u}'' \qquad [1.144]$$

while the geometric boundary conditions [1.46] are:

$$\begin{pmatrix} u_H \\ u'_H \end{pmatrix} = \begin{pmatrix} \breve{u}_H \\ \breve{\theta}_H \end{pmatrix} \qquad [1.145]$$

To build up the extended Green identity for the velocity constraint operator, equation [1.51], we take a dummy vector $\boldsymbol{p}_c = (p_c, c_c)^T$, perform the scalar product $\boldsymbol{p}_c^T \boldsymbol{A} \dot{\boldsymbol{w}}_m = p_c \dot{u} + c_c \dot{u}'$, and integrate by parts over $\mathcal{S}$ to free the velocities from the space-derivatives, thus obtaining:

$$\int_{\mathcal{S}} \left(p_c \dot{u} + c_c \dot{u}'\right) ds = \int_{\mathcal{S}} \dot{u} \left(p_c - c_c'\right) ds + \left[\dot{u} c_c\right]_A^B \qquad [1.146]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Therefore, the equilibrium condensation operators are (check equations [1.52]):

$$\boldsymbol{A}^{\star}=\begin{pmatrix}1 & -\partial_s\end{pmatrix}, \qquad \boldsymbol{\mathcal{A}}_A^{\star}=\begin{pmatrix}0, & -1\end{pmatrix}, \qquad \boldsymbol{\mathcal{A}}_B^{\star}=\begin{pmatrix}0, & 1\end{pmatrix} \qquad [1.147]$$

Note that $\boldsymbol{A}^{\star}\boldsymbol{D}_c^{\star}=-\partial_s+\partial_s=0$.

To condense the equilibrium equations, we could use the previous operators directly in equations [1.54] and [1.55]. However, for illustrative purposes, we restart the whole procedure. First, we write the VPP in the form [1.48], by using $\dot{\kappa}=\dot{\theta}'$:

$$\begin{aligned}&\int M\dot{\theta}'ds=\int_{\mathcal{S}}\left(p\dot{u}+c\dot{\theta}\right)ds+P_A\dot{u}_A+P_B\dot{u}_B+C_A\dot{\theta}_A+C_B\dot{\theta}_B\\&\quad\forall\left(\dot{u},\dot{\theta}\right)|\dot{\theta}=\dot{u}'\end{aligned} \qquad [1.148]$$

Note that *the constraint has not been substituted, yet*! Then, we perform a first integration by parts:

$$\begin{aligned}&\int_{\mathcal{S}}\left[-p\dot{u}-\left(M'+c\right)\dot{\theta}\right]ds-P_A\dot{u}_A-P_B\dot{u}_B\\&\quad-\left(M_A+C_A\right)\dot{\theta}_A-\left(-M_B+C_B\right)\dot{\theta}_B=0\qquad\forall\left(\dot{u},\dot{\theta}\right)|\dot{\theta}=\dot{u}'\end{aligned} \qquad [1.149]$$

Only after that, we substitute the constraint:

$$\begin{aligned}&\int_{\mathcal{S}}\left[-p\dot{u}-\left(M'+c\right)\dot{u}'\right]ds-P_A\dot{u}_A-P_B\dot{u}_B\\&\quad-\left(M_A+C_A\right)\dot{u}_A'-\left(-M_B+C_B\right)\dot{u}_B'=0\qquad\forall\dot{u}'\end{aligned} \qquad [1.150]$$

This equation is in the form of equation [1.50], where the first two boundary terms refer to the master variable, and the last two to the slave variable, expressed in terms of the master one. A second integration by parts leads to:

$$\begin{aligned}&\int_{\mathcal{S}}\left[-p+\left(M'+c\right)'\right]\dot{u}ds-\left[(M'+c)\dot{u}\right]_A^B-P_A\dot{u}_A-P_B\dot{u}_B\\&\quad-\left(M_A+C_A\right)\dot{u}_A'-\left(-M_B+C_B\right)\dot{u}_B'=0\qquad\forall\dot{u}'\end{aligned} \qquad [1.151]$$

Because of the arbitrariness of $\dot{u}$, we get:

$$\begin{aligned}&M''=p-c'\\&(M_A'+c_A-P_A)\dot{u}_A=0,\qquad\left(-M_A-C_A\right)\dot{u}_A'=0\\&(-M_B'-c_B-P_B)\dot{u}_B=0,\quad\left(M_B-C_B\right)\dot{u}_B'=0\end{aligned} \qquad [1.152]$$

Therefore, the couple density c contributes to the translational equilibrium, in the field and at the boundaries (i.e. they enter the “master part” of the equation, not the “slave part”, as stated by equations [1.56] and [1.96]).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

If, e.g., the beam is clamped at A ($\dot{u}_A = 0,\ \dot{u}'_A = 0$) and free at B ($\dot{u}_B \neq 0,\ \dot{u}'_B \neq 0$), the mechanical conditions are $-\left(M'_B + c_B + P_B\right) = 0$, $M_B - C_B = 0$. By using kinematics and the elastic law, we have $M = EJu''$, from which:

$$
\begin{aligned}
& EJu'''' = p - c' \\
& u_A = \breve{u}_A, \qquad\qquad\qquad u'_A = \breve{\theta}_A \\
& -\left(EIu'''_B + c_B + P_B\right) = 0, \quad EJu''_B - C_B = 0
\end{aligned}
\qquad [1.153]
$$

Once the problem has been solved, the field balance equation [1.140b] read as equations [1.61a]:

$$
\begin{aligned}
& T' = -p \\
& T = -c - EJu'''
\end{aligned}
\qquad [1.154]
$$

These are *not* independent, because of [1.153a]. From either of them, the reactive stress T is drawn.

1.8 Summary

In this chapter, we formulated a 1D beam metamodel, i.e. an ensemble of property and rules that each specific model, to be developed in the following chapters, must obey. It calls for analyzing: (a) kinematics, (b) dynamics, and (c) rheology of the model.

We started, in section 1.2, analyzing internally unconstrained beams, by defining column-vectors of unknown generalized displacements, strains and stresses, and known generalized field forces and boundary forces (generally non-conservative), as well as boundary displacements. We separately addressed kinematics, dynamics and rheology.

Concerning kinematics, we first discussed *locally rigid/non-rigid beams*, as 1D bodies not-endowed/endowed with kinematic descriptors able to account for the "change of shape" of the point (typically the deformation of the underlying cross-section). We introduced *nonlinear* strain–displacement relationships, whose time-differentiation led to *linear* strain-rate-velocity relationships, which define the *(differential) kinematic operator*. Since this is configuration dependent, it differs from that of the linear theory, which is evaluated at the reference configuration. Differential relationships are sided by algebraic *geometric boundary conditions*, prescribing displacements at the ends of the beam.

Concerning dynamics, we derived balance (or equilibrium) equations, and *mechanical (or natural) boundary conditions*, via VPP. This states that an equality must hold between the powers spent by forces on virtual velocities, on one side, and stresses on virtual strain-rates, on the other side, when arbitrary strain-rates and

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

velocities are assigned to the body, provided they are respectful of the kinematic constraints. The VPP provides differential balance equations, governed by an *equilibrium operator*, and algebraic boundary conditions, both linear in the stresses (but nonlinear in the displacements). The VPP can also be read as an extended Green identity, which states that the equilibrium operator is the *adjoint* of the kinematic operator, and that mechanical boundary conditions are the adjoint of the geometrical boundary conditions. Such a property is known as *duality property;* differently from the linear theory, this holds in the current (not in the reference!) configuration, to which the virtual motion is superimposed. If the beam is locally rigid, the balance equations can also be interpreted (or alternatively derived) as the cardinal equation of motion (or equilibrium) of an infinitesimal segment of the beam, and the mechanical boundary conditions as the equality of the emerging stresses to the forces applied to the boundary.

Concerning rheology, we limited ourselves to *hyperelastic materials* (often called although improperly, elastic), for which stresses at a point at an instant not only depend on strains at the same point at the same instant, as occurs for simply elastic materials, but, moreover, the stresses spend a deformation work over the strains, which is independent of the strain-path. Therefore, hyperelasticity is synonymous of conservativeness (i.e. lack of dissipation) of the material. It entails the existence of an *elastic potential*, function of the strains, from which stresses are derived by differentiation. Here, *linear* hyperelastic materials were considered only, for which stresses and strains are proportional, by the way of an *elastic matrix*.

The equations of the problem were combined according to the *displacement method*, which consists of expressing the balance equations in terms of the displacements only, which are therefore the main unknown of the problem. Linearization of these equations around the reference configuration supplies the familiar *(tangent) stiffness operators* (in the domain and at the boundary) of the linear theory.

In section 1.3 we considered internally constrained beams, in which one or more of the strains are prescribed to identically vanish along the beam. The constraints call for splitting the generalized strain vector into an *unconstrained (or admissible) part*, collecting the non-zero strains, and a *constrained part*, collecting the vanishing strains. Accordingly, the generalized stress vector was split ino the *active part*, and a (maybe, partially) *reactive part*, concerning the stresses spending power on the admissible and constrained strains, respectively. Because of the reactive character of part of the stresses, the elastic law only involves active stresses and admissible strains. The equations of the constrained problem were combined according to two different philosophies: (a) *the mixed formulation*, in which displacements and reactive stresses were assumed as the main variables; and (b) *the displacement formulation*, in which the equations were further manipulated to eliminate reactive stresses.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

In the mixed formulation, the VPP must account for the prescribed internal constraints. These are introduced by the *Lagrange multipliers* technique, which, in the case studied here, just assumes the physical meaning of reactive stresses. The balance equations supplied by the VPP contain active as well reactive stresses, only the former being expressible in terms of displacements via the elastic law and the unconstrained strain–displacement relationships. The increased number of unknowns, however, is balanced by the nonlinear constraint relationships (i.e. the conditions of vanishing of the restrained strains), which must be appended to the balance equations.

The task of the displacement formulation consists of eliminating the reactive stresses from the equation of motion, and, moreover, to express them in terms of a reduced set of *free* displacement variables, able to describe the most general configuration of the body compatible with the constraints. The goal is similar to that of the analytical mechanics, in which we want to write the Lagrange equations of motion in terms of Lagrange parameters only. To this end, the constraint equations are solved (when possible, and maybe by a perturbation method) to express a set of *slave variables* as function of the remaining *master (or free) variables*. The relationship linking all displacements to the master displacements is called the *constraint for displacements*. By using it, the (active) strain–displacement relationships and the geometric boundary conditions are expressed in terms of master variables only, this operation being referred to as the *condensation of the kinematic equations*. When the constraints for displacements are time-differentiated, linear *constraints for velocities* are obtained (although nonlinear in the displacements, since referred to the current configuration). These relationships define a *(differential) velocity constraint operator*, which plays an important role in the formulation. To filter reactive stresses, we used the VPP, in which the velocity constraints were directly substituted (and not accounted for via Lagrange multipliers, as done in the mixed formulation!). The procedure leads to balance equations which are linear combinations of the original equation, able to automatically filter the reactive stresses. The linear operator acting on them is called the *equilibrium condensation operator*, which turns out to be the adjoint of the velocity constraint operator. When we combine the condensed kinematic and equilibrium equations, and we make use of the elastic law, final equilibrium equations, pure in the master variables, are obtained. Reactive stresses, if of interest, can be derived after having solved the elastic problem, by resorting to the non-condensed balance equations. Although they appear in an over-determined form, they can be solved, since the relevant compatibility condition is satisfied by the VPP itself!

In sections 1.4 and 1.5, we studied *prestressed beams*. These are bodies subjected to time-independent preloads which bring the beam into a prestressed configuration, which is taken as reference configuration, in place of the natural one. After that, incremental loads, possibly time-dependent, act on the beam, by bringing it into the current configuration. The main difference with the formulation of the stress-free

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

beams relies on the elastic law, which becomes linear but non-homogeneous, to account for prestresses when the incremental strains are zero. To simplify the analysis, prestrains are usually neglected, i.e. the beam is assumed to undergo a prestress by keeping its original geometry. For these beams the linear approximation (according to the *linearized theory*) is of remarkable importance in the technical applications, since it allows us to solve important problems such as: to find the critical value of the load in buckling problems; to evaluate the eigenfrequencies of strings and cables; to determine the response of prestressed beam/cables to *small incremental loads*, and/or imperfections; i.e. solving *linear* problems in which, however, the *geometric stiffness*, related to the prestress, plays a non-negligible role. If the beam is internally unconstrained, the prestress simply adds an extra-term to the nonlinear equilibrium equation, with respect to the unprestressed case. If, in contrast, the beam is internally constrained, we have to distinguish: (a) in the mixed formulation, the incremental reactive stress also appears among the unknowns, while the prestress contributes to the stiffness of the beam; and (b) in the displacement formulation, all the reactive stresses, pre-existing and incremental, are filtered, so that only the active prestress appears in the stiffness.

In closing the chapter, all the previous models were reformulated by an alternative approach, the *variational formulation*. This consists of enforcing the stationary condition of the TPE functional, over the domain of the admissible displacements. Internal constraints can also be taken into account by introducing Lagrange multipliers. The approach only requires analyzing kinematics and elasticity, and furnishes the balance equations directly in terms of displacements, and, possibly, reactive stresses. As a drawback, it can only be used for conservative forces. Remarkably, its varied form is just the virtual work principle, which in contrast holds for any type of force. The linearized theory also admits a variational formulation, when the forces are conservative, in which only the second-order part of the TPE is retained. In particular, the geometric stiffness comes out of the work spent by the prestresses in the second-order part of the strains.

Tables 1.1 and 1.2 summarize the main results of the analysis carried out in the chapter. They report the solving equations for (a) the nonlinear Fundamental Problem, (b) for the linear/linearized problem, and (c) the relevant expressions for the TPE, for all cases examined: unconstrained/constrained, unprestressed/prestressed beams and, when appropriated, mixed/displacement formulations. The tables make it easy to compare formulas, and to appreciate the contributions of reactive stresses and/or prestresses.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Nonlinear Problem	Linear Problem
Unconstrained & Unprestressed beams	
$\boldsymbol{D}^\star\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{p}$ $\boldsymbol{\mathcal{D}}^\star_H\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}_H\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{P}_H$ $\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H$	$\boldsymbol{L}\boldsymbol{w}=\boldsymbol{p}_0$ $\boldsymbol{\mathcal{L}}_H\boldsymbol{w}=\boldsymbol{P}_{0H}$ where: $\boldsymbol{L}:=\boldsymbol{D}^\star_0\boldsymbol{E}\boldsymbol{D}_0,\quad \boldsymbol{\mathcal{L}}_H:=\boldsymbol{\mathcal{D}}^\star_{0H}\boldsymbol{E}\boldsymbol{D}_{0H}$
Constrained & Unprestressed beams: Mixed Formulation	
$\boldsymbol{D}^\star_u\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\boldsymbol{D}^\star_c\boldsymbol{\sigma}_c=\boldsymbol{p}$ $\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w},\boldsymbol{w}'\right)=\boldsymbol{0}$ $\boldsymbol{\mathcal{D}}^\star_{uH}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_{uH}+\boldsymbol{\mathcal{D}}^\star_{cH}\boldsymbol{\sigma}_c=\boldsymbol{P}_H$ $\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H$	$\begin{pmatrix}\boldsymbol{L}_u & \boldsymbol{D}^\star_{0c}\\ \boldsymbol{D}_{0c} & \boldsymbol{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{w}\\ \boldsymbol{\sigma}_c\end{pmatrix}=\begin{pmatrix}\boldsymbol{p}_0\\ \boldsymbol{0}\end{pmatrix}$ $\boldsymbol{\mathcal{L}}_{uH}\boldsymbol{w}+\boldsymbol{\mathcal{D}}^\star_{0cH}\boldsymbol{\sigma}_c=\boldsymbol{P}_{0H}$ $\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H$ where: $\boldsymbol{L}_u:=\boldsymbol{D}^\star_{0u}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u},\quad \boldsymbol{\mathcal{L}}_{uH}:=\left[\boldsymbol{\mathcal{D}}^\star_{0u}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}\right]_H$
Constrained & Unprestressed beams: Displacement Formulation	
$\boldsymbol{A}^\star\boldsymbol{D}^\star_u\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u=\boldsymbol{A}^\star\boldsymbol{p}$ $\left[\boldsymbol{\mathcal{A}}^\star\boldsymbol{D}^\star_u\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u+\boldsymbol{\mathcal{D}}^\star_{um}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\right]_H=\left[\boldsymbol{P}_m+\boldsymbol{\mathcal{A}}^\star\boldsymbol{p}\right]_H$ $\left[\boldsymbol{\mathcal{D}}^\star_{us}\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\right]_H=\left[\boldsymbol{P}_s\right]_H$ $\boldsymbol{w}_{mH}=\breve{\boldsymbol{w}}_{mH},\quad \boldsymbol{\mathcal{W}}_{sH}\left(\boldsymbol{w}_m,\boldsymbol{w}'_m,\ldots\right)=\breve{\boldsymbol{w}}_{sH}$	$\boldsymbol{A}^\star_0\boldsymbol{L}_u\boldsymbol{A}_0\boldsymbol{w}_m=\boldsymbol{A}^\star_0\boldsymbol{p}_0$ $\left[\boldsymbol{\mathcal{A}}^\star_0\boldsymbol{L}_u\boldsymbol{A}_0+\boldsymbol{\mathcal{L}}_{um}\boldsymbol{A}_0\right]_H\boldsymbol{w}_m=\left[\boldsymbol{P}_{0m}+\boldsymbol{\mathcal{A}}^\star_0\boldsymbol{p}_0\right]_H$ $\left[\boldsymbol{\mathcal{L}}_{us}\boldsymbol{A}_0\right]_H\boldsymbol{w}_m=\left[\boldsymbol{P}_{0s}\right]_H$ $\boldsymbol{w}_{mH}=\breve{\boldsymbol{w}}_{mH},\quad \left[\boldsymbol{A}_{0s}\boldsymbol{w}_m\right]_H=\breve{\boldsymbol{w}}_{sH}$ where: $\boldsymbol{L}_u:=\boldsymbol{D}^\star_{0u}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u},\ \boldsymbol{\mathcal{L}}_{umH}:=\left[\boldsymbol{\mathcal{D}}^\star_{0um}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}\right]_H$ $\boldsymbol{\mathcal{L}}_{usH}:=\left[\boldsymbol{\mathcal{D}}^\star_{0us}\boldsymbol{E}_{uu}\boldsymbol{D}_{0u}\right]_H$
Unconstrained & Prestressed beams	
Nonlinear Problem	Linearized Problem
$\boldsymbol{D}^\star\boldsymbol{E}\boldsymbol{\mathcal{E}}+\left(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}}-\mathring{\boldsymbol{p}}\right)=\tilde{\boldsymbol{p}}$ $\boldsymbol{\mathcal{D}}^\star_H\boldsymbol{E}\boldsymbol{\mathcal{E}}_H+\left(\boldsymbol{\mathcal{D}}^\star_H\mathring{\boldsymbol{\sigma}}-\mathring{\boldsymbol{P}}_H\right)=\tilde{\boldsymbol{P}}_H$ $\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H$	$\boldsymbol{L}\boldsymbol{w}+\boldsymbol{G}\boldsymbol{w}=\tilde{\boldsymbol{p}}_0$ $\boldsymbol{\mathcal{L}}_H\boldsymbol{w}+\boldsymbol{\mathcal{G}}_H\boldsymbol{w}=\tilde{\boldsymbol{P}}_0$ $\boldsymbol{w}_H=\breve{\boldsymbol{w}}_H$ where: $\boldsymbol{L}:=\boldsymbol{D}^\star_0\boldsymbol{E}\boldsymbol{D}_0,\quad \boldsymbol{\mathcal{L}}_H:=\boldsymbol{\mathcal{D}}^\star_{0H}\boldsymbol{E}\boldsymbol{D}_{0H}$ $\boldsymbol{G}:=\left(\frac{\partial(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}})}{\partial\boldsymbol{w}}\right)_0+\left(\frac{\partial(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}})}{\partial\boldsymbol{w}'}\right)_0\frac{\partial}{\partial s}-\left(\frac{\partial\mathring{\boldsymbol{p}}}{\partial\boldsymbol{w}}\right)_0$ $\boldsymbol{\mathcal{G}}_H:=\left(\frac{\partial(\boldsymbol{\mathcal{D}}^\star_H\mathring{\boldsymbol{\sigma}})}{\partial\boldsymbol{w}}\right)_0+\left(\frac{\partial(\boldsymbol{\mathcal{D}}^\star_H\mathring{\boldsymbol{\sigma}})}{\partial\boldsymbol{w}'}\right)_0\frac{\partial}{\partial s}-\left(\frac{\partial\mathring{\boldsymbol{P}}_H}{\partial\boldsymbol{w}}\right)_0$

Table 1.1: The Fundamental Problem: nonlinear and linear/linearized equations

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Constrained & Prestressed beams: Mixed Formulation	
Nonlinear Problem	Linearized Problem
$\boldsymbol{D}_u^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{D}_c^\star\tilde{\boldsymbol{\sigma}}_c + (\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{p}}) = \tilde{\boldsymbol{p}}$ $\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w}, \boldsymbol{w}'\right) = \boldsymbol{0}$ $\boldsymbol{\mathcal{D}}_{uH}^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_{uH} + \boldsymbol{\mathcal{D}}_{cH}^\star\tilde{\boldsymbol{\sigma}}_c + \left(\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{P}}_H\right) = \tilde{\boldsymbol{P}}_H$ $\boldsymbol{w}_H = \breve{\boldsymbol{w}}_H$	$\left(\begin{pmatrix}\boldsymbol{L}_u & \boldsymbol{D}_{0c}^\star \\ \boldsymbol{D}_{0c} & \boldsymbol{0}\end{pmatrix} + \begin{pmatrix}\boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}\end{pmatrix}\right)\begin{pmatrix}\boldsymbol{w} \\ \tilde{\boldsymbol{\sigma}}_c\end{pmatrix} = \begin{pmatrix}\tilde{\boldsymbol{p}}_0 \\ \boldsymbol{0}\end{pmatrix}$ $\boldsymbol{\mathcal{L}}_{uH}\boldsymbol{w} + \boldsymbol{\mathcal{G}}_H\boldsymbol{w} + \boldsymbol{\mathcal{D}}_{0cH}^\star\tilde{\boldsymbol{\sigma}}_c = \tilde{\boldsymbol{P}}_{0H}$ $\boldsymbol{w}_H = \breve{\boldsymbol{w}}_H$ where: $\boldsymbol{L}_u := \boldsymbol{D}_{0u}^\star \boldsymbol{E}_{uu}\boldsymbol{D}_{0u}$, $\boldsymbol{\mathcal{L}}_{uH} := \left[\boldsymbol{\mathcal{D}}_{0u}^\star \boldsymbol{E}_{uu}\boldsymbol{D}_{0u}\right]_H$ $\boldsymbol{G} := \left(\frac{\partial(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}})}{\partial\boldsymbol{w}}\right)_0 + \left(\frac{\partial(\boldsymbol{D}^\star\mathring{\boldsymbol{\sigma}})}{\partial\boldsymbol{w}'}\right)_0\frac{\partial}{\partial s} - \left(\frac{\partial\mathring{\boldsymbol{p}}}{\partial\boldsymbol{w}}\right)_0$ $\boldsymbol{\mathcal{G}}_H := \left(\frac{\partial(\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}})}{\partial\boldsymbol{w}}\right)_0 + \left(\frac{\partial(\boldsymbol{\mathcal{D}}_H^\star\mathring{\boldsymbol{\sigma}})}{\partial\boldsymbol{w}'}\right)_0\frac{\partial}{\partial s} - \left(\frac{\partial\mathring{\boldsymbol{P}}_H}{\partial\boldsymbol{w}}\right)_0$

Constrained & Prestressed beams: Displacement Formulation	
Nonlinear Problem	Linearized Problem
$\boldsymbol{A}^\star\boldsymbol{D}_u^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{A}^\star\left(\boldsymbol{D}_u^\star\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{p}}\right) = \boldsymbol{A}^\star\tilde{\boldsymbol{p}}$ $\Big[\boldsymbol{\mathcal{A}}^\star\boldsymbol{D}_u^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{\mathcal{D}}_{um}^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \boldsymbol{\mathcal{A}}^\star\left(\boldsymbol{D}_u^\star\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{p}}\right)$ $+ \left(\boldsymbol{\mathcal{D}}_{um}^\star\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}_m\right)\Big]_H = \left[\tilde{\boldsymbol{P}}_m + \boldsymbol{\mathcal{A}}^\star\tilde{\boldsymbol{p}}\right]_H$ $\left[\boldsymbol{\mathcal{D}}_{us}^\star \boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u + \left(\boldsymbol{\mathcal{D}}_{us}^\star\mathring{\boldsymbol{\sigma}}_u - \mathring{\boldsymbol{P}}_s\right)\right]_H = \left[\tilde{\boldsymbol{P}}_s\right]_H$ $\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad \boldsymbol{\mathcal{W}}_{sH}\left(\boldsymbol{w}_m, \boldsymbol{w}_m', \ldots\right) = \breve{\boldsymbol{w}}_{sH}$	$\boldsymbol{A}_0^\star\boldsymbol{L}_u\boldsymbol{A}_0\boldsymbol{w}_m + \boldsymbol{A}_0^\star\boldsymbol{G}_u\boldsymbol{A}_0\boldsymbol{w}_m = \boldsymbol{A}_0^\star\tilde{\boldsymbol{p}}_0$ $\left[\boldsymbol{\mathcal{A}}_0^\star\boldsymbol{L}_u\boldsymbol{A}_0 + \boldsymbol{\mathcal{L}}_{um}\boldsymbol{A}_0 + \boldsymbol{\mathcal{A}}_0^\star\boldsymbol{G}_u\boldsymbol{A}_0 + \boldsymbol{\mathcal{G}}_{um}\boldsymbol{A}_0\right]_H\boldsymbol{w}_m$ $= \left[\tilde{\boldsymbol{P}}_{0m} + \boldsymbol{\mathcal{A}}_0^\star\tilde{\boldsymbol{p}}_0\right]_H$ $\left[\boldsymbol{\mathcal{L}}_{us}\boldsymbol{A}_0 + \boldsymbol{\mathcal{G}}_{us}\boldsymbol{A}_0\right]_H\boldsymbol{w}_m = \left[\tilde{\boldsymbol{P}}_{0s}\right]_H$ $\boldsymbol{w}_{mH} = \breve{\boldsymbol{w}}_{mH}, \quad \left[\boldsymbol{A}_{0s}\boldsymbol{w}_m\right]_H = \breve{\boldsymbol{w}}_{sH}$ where: $\boldsymbol{L}_u := \boldsymbol{D}_{0u}^\star \boldsymbol{E}_{uu}\boldsymbol{D}_{0u}$, $\boldsymbol{\mathcal{L}}_{umH} := \left[\boldsymbol{\mathcal{D}}_{0um}^\star \boldsymbol{E}_{uu}\boldsymbol{D}_{0u}\right]_H$ $\boldsymbol{\mathcal{L}}_{usH} := \left[\boldsymbol{\mathcal{D}}_{0us}^\star \boldsymbol{E}_{uu}\boldsymbol{D}_{0u}\right]_H$ $\boldsymbol{G}_u := \left(\frac{\partial(\boldsymbol{D}_u^\star\mathring{\boldsymbol{\sigma}}_u)}{\partial\boldsymbol{w}}\right)_0$ $+ \left(\frac{\partial(\boldsymbol{D}_u^\star\mathring{\boldsymbol{\sigma}}_u)}{\partial\boldsymbol{w}'}\right)_0\frac{\partial}{\partial s} - \left(\frac{\partial\mathring{\boldsymbol{p}}}{\partial\boldsymbol{w}}\right)_0$ $\boldsymbol{\mathcal{G}}_{uH} := \left(\frac{\partial(\boldsymbol{\mathcal{D}}_{uH}^\star\mathring{\boldsymbol{\sigma}}_u)}{\partial\boldsymbol{w}}\right)_0 + \left(\frac{\partial(\boldsymbol{\mathcal{D}}_{uH}^\star\mathring{\boldsymbol{\sigma}}_u)}{\partial\boldsymbol{w}'}\right)_0\frac{\partial}{\partial s}$ $- \left(\frac{\partial\mathring{\boldsymbol{P}}_H}{\partial\boldsymbol{w}}\right)_0 + \ldots$

Table 1.1: (Continued) The Fundamental Problem: nonlinear and linear/linearized equations

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Unconstrained & Unprestressed beams	
Nonlinear Problem	Linear Problem
$\Pi\left[\boldsymbol{w}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$ $-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H$	$\Pi\left[\boldsymbol{w}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}^{(1)T}\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}^{(1)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$ $-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H$ where: $\boldsymbol{\mathcal{E}}^{(1)} = \boldsymbol{D}_0\boldsymbol{w}$

Constrained & Unprestressed beams: Mixed Formulation	
Nonlinear Problem	Linear Problem
$\breve{\Pi}\left[\boldsymbol{w},\boldsymbol{\lambda}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$ $-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H + \int_{\mathcal{S}}\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$	$\breve{\Pi}\left[\boldsymbol{w},\boldsymbol{\lambda}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^{(1)T}\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u^{(1)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$ $-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H + \int_{\mathcal{S}}\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c^{(1)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$ where: $\boldsymbol{\mathcal{E}}_u^{(1)} = \boldsymbol{D}_{0u}\boldsymbol{w}, \quad \boldsymbol{\mathcal{E}}_c^{(1)} = \boldsymbol{D}_{0c}\boldsymbol{w}$

Constrained & Unprestressed beams: Displacement Formulation	
Nonlinear Problem	Linear Problem
$\Pi_u\left[\boldsymbol{w}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$ $-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H$ $\boldsymbol{w} = \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)$	$\Pi_u\left[\boldsymbol{w}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}_u^{(1)T}\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u^{(1)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$ $-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H, \quad \boldsymbol{w} = \boldsymbol{A}_0\boldsymbol{w}_m$ where: $\boldsymbol{\mathcal{E}}_u^{(1)} = \boldsymbol{D}_{0u}\boldsymbol{w}$

Unconstrained & Prestressed beams	
Nonlinear Problem	Linearized Problem
$\Pi\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] := \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{\mathcal{E}}^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}\boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$ $-\int_{\mathcal{S}} \boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H + \int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}^T\boldsymbol{\mathcal{E}}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$ $-\int_{\mathcal{S}} \boldsymbol{w}^T\mathring{\boldsymbol{p}}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\mathring{\boldsymbol{P}}_H$	$\Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] := \int_{\mathcal{S}}\left(\frac{1}{2}\boldsymbol{\mathcal{E}}^{(1)T}\boldsymbol{E}\boldsymbol{\mathcal{E}}^{(1)} + \mathring{\boldsymbol{\sigma}}^T\boldsymbol{\mathcal{E}}^{(2)}\right)ds$ $-\int_{\mathcal{S}} \boldsymbol{w}^T\tilde{\boldsymbol{p}}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\tilde{\boldsymbol{P}}_H$

Table 1.2: The Variational formulation: the EPT functional for nonlinear and linear/linearized theories

<table>
<tr><th colspan="2">Constrained & Prestressed beams: Mixed Formulation</th></tr>
<tr><td>Nonlinear Problem</td><td>Linearized Problem</td></tr>
<tr><td>$$\check{\Pi}\left[\boldsymbol{w},\boldsymbol{\lambda};\mathring{\boldsymbol{\sigma}}_u\right] := \frac{1}{2}\int_{\mathcal{S}}\boldsymbol{\mathcal{E}}_u^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$$ $$-\int_{\mathcal{S}}\boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H + \int_{\mathcal{S}}\boldsymbol{\lambda}^T\boldsymbol{\mathcal{E}}_c\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$$ $$+\int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}_u^T\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds - \int_{\mathcal{S}}\boldsymbol{w}^T\mathring{\boldsymbol{p}}ds$$ $$-\sum_{H=A}^{B}\boldsymbol{w}_H^T\mathring{\boldsymbol{P}}_H$$</td><td>$$\check{\Pi}^{(2)}\left[\boldsymbol{w},\tilde{\boldsymbol{\sigma}}_c;\mathring{\boldsymbol{\sigma}}\right] := \int_{\mathcal{S}}\boldsymbol{\mathcal{E}}_u^{(1)T}(\boldsymbol{w},\boldsymbol{w}')\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u^{(1)}(\boldsymbol{w},\boldsymbol{w}')ds$$ $$-\int_{\mathcal{S}}\boldsymbol{w}^T\tilde{\boldsymbol{p}}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\tilde{\boldsymbol{P}}_H$$ $$+\int_{\mathcal{S}}\left(\mathring{\boldsymbol{\sigma}}^T\boldsymbol{\mathcal{E}}^{(2)} + \tilde{\boldsymbol{\sigma}}_c\boldsymbol{\mathcal{E}}_c^{(1)}\right)ds$$</td></tr>
</table>

<table>
<tr><th colspan="2">Constrained & Prestressed beams: Displacement Formulation</th></tr>
<tr><td>Nonlinear Problem</td><td>Linearized Problem</td></tr>
<tr><td>$$\Pi\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}\right] := \frac{1}{2}\int_{\mathcal{S}}\boldsymbol{\mathcal{E}}_u^T\left(\boldsymbol{w},\boldsymbol{w}'\right)\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$$ $$-\int_{\mathcal{S}}\boldsymbol{w}^T\boldsymbol{p}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\boldsymbol{P}_H$$ $$+\int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}_u^T\boldsymbol{\mathcal{E}}_u\left(\boldsymbol{w},\boldsymbol{w}'\right)ds - \int_{\mathcal{S}}\boldsymbol{w}^T\mathring{\boldsymbol{p}}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\mathring{\boldsymbol{P}}_H$$ $$\boldsymbol{w} = \boldsymbol{\mathcal{W}}\left(\boldsymbol{w}_m,\boldsymbol{w}_m',\ldots\right)$$</td><td>$$\Pi^{(2)}\left[\boldsymbol{w};\mathring{\boldsymbol{\sigma}}_u\right] := \int_{\mathcal{S}}\boldsymbol{\mathcal{E}}_u^{(1)T}(\boldsymbol{w},\boldsymbol{w}')\boldsymbol{E}_{uu}\boldsymbol{\mathcal{E}}_u(\boldsymbol{w},\boldsymbol{w}')ds$$ $$-\int_{\mathcal{S}}\boldsymbol{w}^T\tilde{\boldsymbol{p}}ds - \sum_{H=A}^{B}\boldsymbol{w}_H^T\tilde{\boldsymbol{P}}_H + \int_{\mathcal{S}}\mathring{\boldsymbol{\sigma}}_u^T\boldsymbol{\mathcal{E}}_u^{(2)}\left(\boldsymbol{w},\boldsymbol{w}'\right)ds$$ $$\boldsymbol{w} = \boldsymbol{A}_0\boldsymbol{w}_m$$</td></tr>
</table>

Table 1.2: (Continued) The Variational formulation: the EPT functional for nonlinear and linear/ linearized theories

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Chapter 2

Straight Beams

We consider a straight beam embedded in a three-dimensional (3D)-space, modeled as a one-dimensional (1D) polar continuum. The kinematics of the beam is first analyzed, by introducing important concepts such as: translations and finite rotations of the body-points; current and reference strains and curvatures; and velocity, spin, velocity gradients and strain-rates. The dynamics of the beam is successively addressed by first applying the Virtual Power Principle (VPP) as a tool for introducing stresses and to provide balance equations and boundary conditions. Then, the problem is approached in an alternate way, in which force-stress and couple-stress acting at a body-point are defined, and then the principles of linear and angular momentum are invoked. Several forms of the scalar balance equations are discussed, namely, in the reference, in the current and in a non-orthogonal basis, the latter naturally stemming from a Lagrangian approach. Constitutive laws are successively formulated, both for hyperelastic and viscoelastic materials, by accounting for possible prestress. An approximate nonlinear elastic law, accounting for large twist, is derived. A brief sketch of homogenization problems is given for beam-like structures. To sum up, all the basic equations are combined to formulate the Fundamental Problem. Finally, the whole formulation is retraced and specialized for the case of planar beams.

2.1 Kinematics

2.1.1 The displacement and rotation fields

Let us consider a prismatic straight beam embedded in a 3D environment space. We define a *centerline*, or beam axis, as the geometrical locus of the centers of area,

Mathematical Models of Beams and Cables Angelo Luongo and Daniele Zulli
© 2013 ISTE Ltd. Published 2013 by ISTE Ltd.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

or *centroids* P, of the cross-sections. We assume that the centerline is flexible, while the cross-sections are rigid, and free to rotate around any axes passing through P. Thus, the 3D-continuum is reduced to a flexible line and a collection of rigid planes connected to it. This object is modeled as a 1D polar continuum, i.e. constituted by *body-points* P endowed with orientation (also called "endowed with a local rigid structure"), able to account for the attitude of the cross-sections (Figure 2.1).

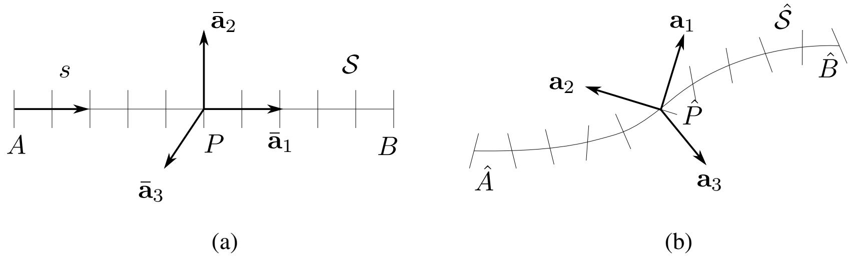


Figure 2.1: Straight beam in 3D and directors: (a) reference configuration; (b) current configuration.

In the reference configuration, the beam axis is assumed to be straight, lying on a segment $\mathcal{S}$ of the space, of extreme-points A, B (Figure 2.1(a)). An abscissa $s \in [0, l]$ is taken on it, with l as the initial beam-length. In the framework of the referential description of motion, we assume s as a label that identifies the generic body-point P, and consequently, we will indifferently use s or P to denote a material point. In this configuration, the cross-sections are assumed to be orthogonal to the axis. To describe their attitude, we rigidly apply *three directors* that form a right orthogonal basis $\bar{\mathcal{B}} := (\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3)$, in which $\bar{\mathbf{a}}_1$ is a unit vector normal to the section (and therefore tangent to $\mathcal{S}$), and $\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3$ are unit vectors lying in the plane of the section, freely chosen, but independent of s.

Displacement and rotation

In the current configuration, occupied by the beam at the time t, the axis is no longer straight, but lies on a (sufficiently smooth) curve $\hat{\mathcal{S}}$, whose regularity properties will be determined later (Figure 2.1(b)). Let $\bar{\mathbf{x}}(s)$ and $\mathbf{x}(s,t)$ be the positions of the point P in the reference and current configurations, respectively, both measured with respect to an arbitrary pole O (Figure 2.2). We define:

$$\mathbf{u} := \mathbf{x}(s,t) - \bar{\mathbf{x}}(s) \quad [2.1]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

to be the *displacement* of the point P at the time t. By introducing scalar components in the basis $\bar{\mathcal{B}}$, we have[1]:

$$\mathbf{u} := u_1(s,t)\bar{\mathbf{a}}_1 + u_2(s,t)\bar{\mathbf{a}}_2 + u_3(s,t)\bar{\mathbf{a}}_3 \tag{2.2}$$

In the current configuration, the triad of directors, solid with the cross-section, forms a basis $\mathcal{B} := (\mathbf{a}_1(s,t), \mathbf{a}_2(s,t), \mathbf{a}_3(s,t))$, which depends on s (in addition to t), and where $\mathbf{a}_j$, $j = 1, 2, 3$, are the new directions of $\bar{\mathbf{a}}_j$. The transformation leading $\bar{\mathbf{a}}_j$ to match $\mathbf{a}_j(s,t)$ is a *rotation* expressed by:

$$\mathbf{a}_j(s,t) = \mathbf{R}(s,t)\bar{\mathbf{a}}_j(s) \tag{2.3}$$

where $\mathbf{R}(s,t)$ is a proper orthogonal *rotation tensor*, meaning that $\det\mathbf{R} = 1$ and $\mathbf{R}^{-1} = \mathbf{R}^T$, i.e.:

$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I} \tag{2.4}$$

The rotation depends on three scalar functions, $\theta_j(s,t)$, $j = 1, 2, 3$, as will be discussed soon.

In summary, the displacement field $\mathbf{u}(s,t)$ describes the translation of the body-points, and therefore determines the current shape $\hat{\mathcal{S}}$ of the centerline; the rotation field $\mathbf{R}(s,t)$ describes the cross-section attitudes. Therefore, any geometric transformation undergone by the beam depends on a vector and a tensor field, or, equivalently, by six scalar fields, $u_j(s,t)$, $\theta_j(s,t)$, $j = 1, 2, 3$, called the *configuration variables*.

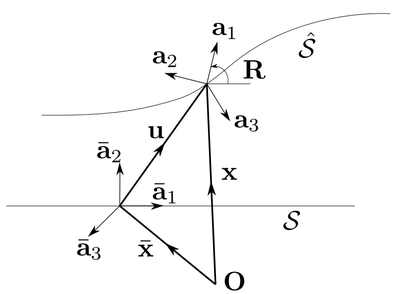


Figure 2.2: Displacement and rotation fields.

1. As a general rule, we will denote by an overbar the components of a vector in $\bar{\mathcal{B}}$, and no overbar for its components in $\mathcal{B}$, for example $\boldsymbol{\omega} = \sum_{i=1}^{n} \bar{\omega}_i \bar{\mathbf{a}}_i = \sum_{i=1}^{n} \omega_i \mathbf{a}_i$. However, to simplify the notation, we will omit the overbar on the components of vectors, as $\mathbf{u}$, that are *always* evaluated in $\bar{\mathcal{B}}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Building the rotation tensor

To build up the tensor $\mathbf{R}$, we compose three successive elementary rotations defined as follows (see Figure 2.3):

1) A rotation $\mathbf{R}_3$ of amplitude θ_3, around $\bar{\mathbf{a}}_3$, which leads the basis $\bar{\mathcal{B}} := (\bar{\mathbf{a}}_j)$ onto the new basis $\check{\mathcal{B}} := (\check{\mathbf{a}}_j)$ (with $\check{\mathbf{a}}_3 \equiv \bar{\mathbf{a}}_3$).

2) A rotation $\mathbf{R}_2$ of amplitude θ_2, around $\check{\mathbf{a}}_2$, which leads the basis $\check{\mathcal{B}}$ onto the new basis $\tilde{\mathcal{B}} := (\tilde{\mathbf{a}}_j)$ (with $\tilde{\mathbf{a}}_2 \equiv \check{\mathbf{a}}_2$).

3) A rotation $\mathbf{R}_1$ of amplitude θ_1, around $\tilde{\mathbf{a}}_1$, which leads the basis $\tilde{\mathcal{B}}$ onto the current basis $\mathcal{B} := (\mathbf{a}_j)$ (with $\mathbf{a}_1 \equiv \tilde{\mathbf{a}}_1$).

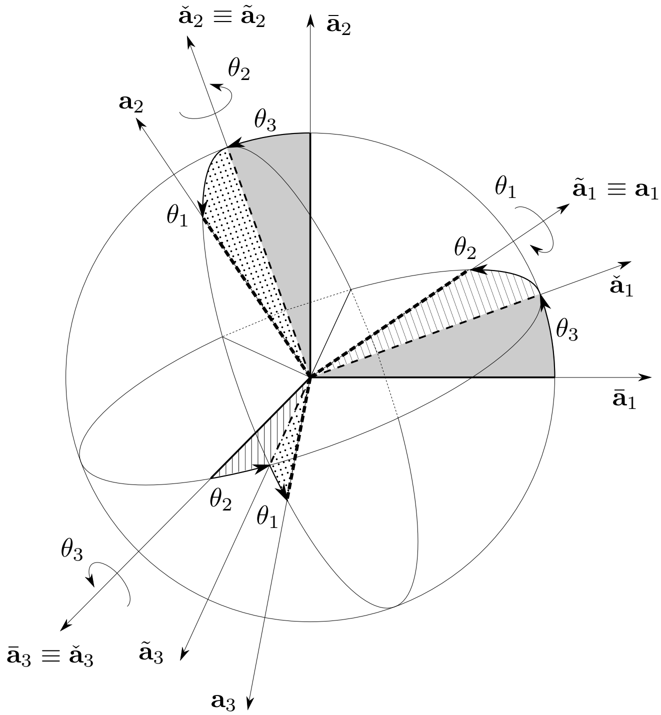


Figure 2.3: Composition of elementary rotations.

Therefore:

$$\mathbf{R} = \mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3 \qquad [2.5]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By evaluating the components of $\mathbf{R}$ in the basis $\bar{\mathcal{B}}$, the following matrix is found:

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta_2\cos\theta_3 & \sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 & \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 \\ \cos\theta_2\sin\theta_3 & \sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 \\ -\sin\theta_2 & \sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 \end{bmatrix} \quad [2.6]$$

that will be referred to as the *rotation matrix*; it collects, column-wise, the components of $\mathbf{a}_j$ onto $\bar{\mathcal{B}}$. The three elementary rotations are measured by the angles θ_i, also called the *Tait–Bryan (or Cardan) angles*[2].

To evaluate the matrix $\bar{\boldsymbol{R}}$ that collects the components of the tensor $\boldsymbol{R}$ in $\bar{\mathcal{B}}$, we need the components of the elementary rotations $\bar{\boldsymbol{R}}_i$ in the same basis, i.e. $\bar{\boldsymbol{R}} = \bar{\boldsymbol{R}}_1\bar{\boldsymbol{R}}_2\bar{\boldsymbol{R}}_3$. However, $\mathbf{R}_3, \mathbf{R}_2, \mathbf{R}_1$ are more easily expressed in the bases $\bar{\mathcal{B}}, \check{\mathcal{B}}, \tilde{\mathcal{B}}$, respectively, in which they assume the simple forms:

$$\bar{\boldsymbol{R}}_3 := \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \check{\boldsymbol{R}}_2 := \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$
$$\tilde{\boldsymbol{R}}_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \quad [2.7]$$

On the other hand, we observe that $\bar{\boldsymbol{R}}_3$ is also the matrix of the change of basis from $\bar{\mathcal{B}}$ to $\check{\mathcal{B}}$ (since it collects the components of the unit vectors of $\check{\mathcal{B}}$ in $\bar{\mathcal{B}}$), and, therefore, $\bar{\boldsymbol{R}}_3^{-1} \equiv \bar{\boldsymbol{R}}_3^T$ is the matrix of the reverse change of basis; hence, $\bar{\boldsymbol{R}}_2 = \bar{\boldsymbol{R}}_3\check{\boldsymbol{R}}_2\bar{\boldsymbol{R}}_3^T$. Similarly, $\bar{\boldsymbol{R}}_1 = \left(\bar{\boldsymbol{R}}_3\check{\boldsymbol{R}}_2\right)\tilde{\boldsymbol{R}}_1\left(\bar{\boldsymbol{R}}_3\check{\boldsymbol{R}}_2\right)^T$. In conclusion:

$$\begin{aligned} \bar{\boldsymbol{R}} &= \left[\left(\bar{\boldsymbol{R}}_3\check{\boldsymbol{R}}_2\right)\tilde{\boldsymbol{R}}_1\left(\bar{\boldsymbol{R}}_3\check{\boldsymbol{R}}_2\right)^T\right]\left[\bar{\boldsymbol{R}}_3\check{\boldsymbol{R}}_2\bar{\boldsymbol{R}}_3^T\right]\left[\bar{\boldsymbol{R}}_3\right] \\ &= \bar{\boldsymbol{R}}_3\check{\boldsymbol{R}}_2\tilde{\boldsymbol{R}}_1 \end{aligned} \quad [2.8]$$

By performing the matrix multiplications, and omitting the overbar, we finally obtain equation [2.6].

REMARK 2.1. It is interesting to note that any rotation $\mathbf{R}$ has the same components in the start and end bases. For example $[\mathbf{R}]_{\bar{\mathcal{B}}} \equiv [\mathbf{R}]_{\mathcal{B}}$; as a matter of fact, by virtue of

2. Sometimes, Tait–Bryan or Cardan angles are called *Euler angles*. Nevertheless, rigorously, Tait–Bryan angles act around axes descending from three different axes (e.g. as in the considered case, the rotations act around $\bar{\mathbf{a}}_3, \check{\mathbf{a}}_2, \tilde{\mathbf{a}}_1$, which descend from the three axes $\bar{\mathbf{a}}_3, \bar{\mathbf{a}}_2, \bar{\mathbf{a}}_1$, respectively), while Euler angles act around axes descending from two different axes (an example would be if the rotation were built by means of three angles, around $\bar{\mathbf{a}}_3, \check{\mathbf{a}}_2, \tilde{\mathbf{a}}_3$, descending just from $\bar{\mathbf{a}}_3, \bar{\mathbf{a}}_2$, respectively).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

equation [2.3], $\bar{\mathbf{a}}_i \cdot \mathbf{R}\bar{\mathbf{a}}_j = \mathbf{R}^T\mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{a}_i \cdot \mathbf{R}\mathbf{a}_j$[3]. In contrast, in the construction illustrated above, matrices differ between them, e.g. $\bar{\boldsymbol{R}}_2 \neq \check{\boldsymbol{R}}_2$, since the bases $\bar{\mathcal{B}}$ and $\check{\mathcal{B}}$ are *not* the start and end bases.

REMARK 2.2. In the nonlinear field, a rotation is a tensor, *not* a vector. In the linear theory, in contrast, the rotation amplitudes $\theta_j(s,t),\ j = 1,2,3$ are dealt with as if they were infinitesimal quantities, so that $\cos\theta_j \simeq 1,\ \sin\theta_j \simeq \theta_j$. This entails that, within higher order terms, $\boldsymbol{R} = \boldsymbol{I} + \boldsymbol{\Omega}$, where $\boldsymbol{I}$ is the identity matrix and:

$$\boldsymbol{\Omega} := \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \qquad [2.9]$$

is a skew-symmetric matrix, whose axial vector is $\boldsymbol{\theta} := (\theta_1, \theta_2, \theta_3)^T$ [4]. The latter is referred to (although not properly) as the "rotation vector" of the linear theory. When, later in the book (see equation [2.60], we introduce the spin matrix $\boldsymbol{W}$, we will observe that $\boldsymbol{\Omega} = \boldsymbol{W}dt$, i.e. the infinitesimal rotation vector is instead of an angular velocity vector, multiplied by an infinitesimal interval of time.

2.1.2 Tackling the rotation tensor

Some problems related to the rotation tensor are now addressed. They give answers to the following questions: (a) how to find the Tait–Bryan angles when the rotation tensor is assigned, (b) how to write the rotation tensor when the rotation axis is assigned, (c) how to extract the rotation axis from an assigned rotation tensor. These questions are not merely theoretical, but, in contrast, have some practical relevance when boundary conditions must to be written, as will appear more clear later.

Finding the elementary rotations

In the last subsection, we solved the problem: "given three elementary rotations (or Tait–Bryan angles) θ_i, find the rotation tensor". The constructive procedure led us to

---

3. We remember that if $\mathbf{v} = \mathbf{L}\mathbf{u}$, where $\mathbf{u} = \sum_j u_j\mathbf{a}_j$ and $\mathbf{v} = \sum_j v_j\mathbf{a}_j$, then $v_i = \mathbf{a}_i \cdot \mathbf{v} = \sum_j u_j\mathbf{a}_i \cdot \mathbf{L}\mathbf{a}_j =: \sum_j l_{ij}u_j$, where $l_{ij} := \mathbf{a}_i \cdot \mathbf{L}\mathbf{a}_j$ are the components of the tensor $\mathbf{L}$ in the basis $(\mathbf{a}_i)$. Moreover, the identity $\mathbf{R}\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{R}^T\mathbf{b}$ has been used.

4. The following identity is known from algebra: $\mathbf{W}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x},\ \forall\mathbf{x}$, where the scalar representations of $\mathbf{W}$ and $\boldsymbol{\omega}$ are, respectively:

$$\boldsymbol{W} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

and where $\boldsymbol{\omega}$ is called the *axial vector* of the skew-tensor $\mathbf{W}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

equation [2.6]. Now, we want to solve the inverse problem: "given a scalar representation of a rotation tensor, find the Tait–Bryan angles θ_i". As we know from rigid-body kinematics (see [BRA 02, SLA 99]), this *nonlinear* problem generally does not admit a unique solution, in the sense that there is more than one triplet of angles θ_i, which leads a body from an initial to an assigned final position.

Let us assume that the rotation matrix $\boldsymbol{R} = [R_{ij}]$ is given (equation [2.6]), representing the tensor $\mathbf{R}$ in $\bar{\mathcal{B}}$ and we want to find θ_i. We can derive θ_1 from the ratio R_{32}/R_{33}, θ_2 from the simplest entry R_{31} and, finally, θ_3 from the ratio R_{21}/R_{11}; therefore:

$$\begin{aligned}\tan\theta_1 &= R_{32}/R_{33}\\ \sin\theta_2 &= -R_{31}\\ \tan\theta_3 &= R_{21}/R_{11}\end{aligned} \qquad [2.10]$$

Even if we confine ourselves to the interval $(-\pi, \pi]$, these equations are ambiguous, since, given a solution $(\theta_1, \theta_2, \theta_3)$, *also the triplet* $(\theta_1 + \pi, \pi - \theta_2, \theta_3 + \pi)$ *is a valid solution*. Note that the solutions are only two and no more, since in evaluating the ratios $R_{32}/R_{33}, R_{21}/R_{11}$, we canceled the common factor $\cos\theta_2$; therefore, once a root has been chosen for θ_2, the ambiguity for θ_1 and θ_3 is resolved. However, if we are interested in moderately large rotations, we can assume that all angles range in the interval $(-\pi/2, \pi/2)$ so that:

$$\begin{aligned}\theta_1 &= \arctan\left(R_{32}/R_{33}\right)\\ \theta_2 &= \arcsin\left(-R_{31}\right)\\ \theta_3 &= \arctan\left(R_{21}/R_{11}\right)\end{aligned} \qquad [2.11]$$

where the inverse trigonometric function returns the principal values of the angles.

A singular case, admitting infinite solutions, occurs when $\theta_2 = \pm\pi/2$, for which the element R_{ij} we chose vanished (see [BRA 02, SLA 99]).

Finding or assigning the rotation axis

As equation [2.6] shows, the rotation of the cross-section is defined by three scalar parameters taken there as the Tait–Bryan angles. However, as it is well-known from rigid-body kinematics, a rotation can also be described by its *rotation axis*, viz. $\bar{\mathbf{n}} = n_1\bar{\mathbf{a}}_1 + n_2\bar{\mathbf{a}}_2 + n_3\bar{\mathbf{a}}_3$ (two independent parameters) and the *rotation angle* Θ (third parameter). Therefore, we can ask ourselves which relation holds between the Tait–Bryan angles θ_i and $(\bar{\mathbf{n}}, \Theta)$. In investigating such a relationship, we have to address two related problems, namely:

1) *Direct problem*: given the axis $\bar{\mathbf{n}}$ and the rotation Θ, find the rotation tensor $\mathbf{R}$.

2) *Inverse problem*: given the rotation tensor $\mathbf{R}$, find the rotation axis $\bar{\mathbf{n}}$ and the rotation Θ.

The direct problem is solved by the so-called *Euler–Rodrigues formula*, which states that:

$$\mathbf{R} = \mathbf{I}\cos\Theta + \bar{\mathbf{n}} \otimes \bar{\mathbf{n}}\,(1 - \cos\Theta) + \bar{\mathbf{N}}\sin\Theta \qquad [2.12]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where $\bar{\mathbf{N}}$ is the skew-symmetric tensor whose axial vector is $\bar{\mathbf{n}}$, i.e. $\bar{\mathbf{N}}\mathbf{r} = \bar{\mathbf{n}} \times \mathbf{r}$, $\forall \mathbf{r}$; moreover, $\otimes$ denotes a dyadic or tensor product, i.e. $[\bar{\mathbf{n}} \otimes \bar{\mathbf{n}}] = \bar{\boldsymbol{n}}\bar{\boldsymbol{n}}^T$. When expressed in the $\bar{\mathcal{B}}$-basis, this formula becomes:

$$\begin{aligned} \boldsymbol{R} = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cos\Theta + \\ & + \begin{bmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2^2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3^2 \end{bmatrix} (1 - \cos\Theta) + \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \sin\Theta \end{aligned} \quad [2.13]$$

Once $\boldsymbol{R}$ has been computed, the Tait–Bryan angles are evaluated by equations [2.11]. The proof of the Euler–Rodrigues is constructive, based on geometrical consideration, and can be found in literature (e.g. see [KOK 06]). Here, we will limit ourselves to checking that it works. In order to prove that, indeed, it describes a rotation of axis $\bar{\mathbf{n}}$ and amplitude Θ, we have to show that (a) $\mathbf{R}\bar{\mathbf{n}} = \bar{\mathbf{n}}$, i.e. the rotation leaves the axis unaltered and (b) that $\bar{\mathbf{m}} \cdot \mathbf{R}\bar{\mathbf{m}} = \cos\Theta$, where $\bar{\mathbf{m}}$ is a unit vector orthogonal to $\bar{\mathbf{n}}$. Concerning task (a), it is sufficient to observe that $(\bar{\mathbf{n}} \otimes \bar{\mathbf{n}})\,\bar{\mathbf{n}} = \bar{\mathbf{n}}$ and $\bar{\mathbf{N}}\bar{\mathbf{n}} = \bar{\mathbf{n}} \times \bar{\mathbf{n}} = \mathbf{0}$; concerning task (b), it is sufficient to observe that $(\bar{\mathbf{n}} \otimes \bar{\mathbf{n}})\,\bar{\mathbf{m}} = (\bar{\mathbf{n}} \cdot \bar{\mathbf{m}})\,\bar{\mathbf{n}} = \mathbf{0}$ and that $\bar{\mathbf{N}}\bar{\mathbf{m}} \perp \bar{\mathbf{m}}$; therefore, both properties are verified.

The inverse problem could be easily solved by observing that, because $\mathbf{R}\bar{\mathbf{n}} = \bar{\mathbf{n}}$, *the rotation axis* $\bar{\mathbf{n}}$ *is the eigenvector of* $\mathbf{R}$ associated with the eigenvalue 1 (this stating that rotation does not stretch the axis). Moreover, since the other two eigenvalues are $\exp(\pm i\Theta)$[5], the rotation could also be computed. However, this approach is not computationally convenient, but use of the Euler–Rodrigues formula is still advisable. Accordingly, if the skew part of $\mathbf{R}$ is computed, namely $\mathrm{skw}\mathbf{R} = \left(\mathbf{R} - \mathbf{R}^T\right)/2$, since $\mathbf{I}$ and $\bar{\mathbf{n}} \otimes \bar{\mathbf{n}}$ are symmetric tensors while $\bar{\mathbf{N}}$ is skew-symmetric, it follows:

$$\bar{\mathbf{N}} = \frac{1}{2\sin\Theta}\left(\mathbf{R} - \mathbf{R}^T\right) \quad [2.14]$$

Then, by identifying the three independent components of the two skew-symmetric tensors, we finally have:

$$\begin{aligned} n_1 &= \frac{1}{2\sin\Theta}\left(R_{32} - R_{23}\right) \\ n_2 &= \frac{1}{2\sin\Theta}\left(R_{13} - R_{31}\right) \\ n_3 &= \frac{1}{2\sin\Theta}\left(R_{21} - R_{12}\right) \end{aligned} \quad [2.15]$$

5. Indeed, in the $\left(\bar{\mathbf{l}}, \bar{\mathbf{m}}, \bar{\mathbf{n}}\right)$-basis, the rotation tensor admits the representation:

$$\boldsymbol{R} := \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

From these, if, e.g., $n_3 \neq 0$, the ratios:

$$\frac{n_1}{n_3} = \frac{R_{32} - R_{23}}{R_{21} - R_{12}}, \qquad \frac{n_2}{n_3} = \frac{R_{13} - R_{31}}{R_{21} - R_{12}} \qquad [2.16]$$

are evaluated, and finally the angle Θ is computed.

2.1.3 The geometric boundary conditions

The displacement and rotation fields are not completely free, since, as we already observed, (a) they must be sufficiently regular, and, moreover, (b) they must fulfill prescriptions at the points where the constraints (i.e. the mechanical devices) are applied. These latter prescriptions are called *geometric boundary conditions*. We assume that the constrained points coincide with the ends $H = A, B$ of the beam; moreover, we admit that the constraints are time-dependent.

When one, or both, the ends are *fully restrained*, the geometric boundary conditions assume the following form:

$$\mathbf{u}_H = \breve{\mathbf{u}}_H(t), \qquad \mathbf{R}_H = \breve{\mathbf{R}}_H(t) \qquad H = A, B \qquad [2.17]$$

where the index H denotes evaluation at one end-point, and the curve-bar indicates a known function of time. We will usually refer to this simplest case. However, *partially restrained* ends are also frequent, and some example will be discussed soon.

The previous vector and tensor equations are equivalent to the following six scalar conditions:

$$u_i = \breve{u}_{iH}, \qquad \theta = \breve{\theta}_{iH}, \qquad i = 1, \ldots, 3 \qquad [2.18]$$

where $\breve{\theta}_i$ are the three independent Tait–Bryan angles of $\breve{\mathbf{R}}_H$. In the matrix form, we also have:

$$\boldsymbol{u} = \breve{\boldsymbol{u}}, \qquad \boldsymbol{\theta} = \breve{\boldsymbol{\theta}} \qquad [2.19]$$

where $\boldsymbol{u} = (u_1, u_2, u_3)^T$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$.

For illustrative purposes, some common constraints are considered here. While conditions on the translations are trivial, those on rotations are, generally, more difficult to express, due to the nonlinear nature of the rotation tensor. We have:

– *Free boundary*: no geometric constraints are prescribed.

– *Spherical hinge*: $u_i = \breve{u}_{iH}$ must be prescribed, while the elementary rotations θ_{iH} are arbitrary.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

– *Clamp*: the six conditions [2.18] must be enforced. If the prescribed rotation $\breve{\Theta}_H$ is zero, then, trivially, the three Tait–Bryan angles $\breve{\theta}_{iH}$ are all zero; if, in contrast, $\breve{\Theta}_H \neq 0$ around a selected rotation axis $\bar{\mathbf{n}}$, then the Euler–Rodrigues formula [2.13] must be used to evaluate $\breve{\mathbf{R}}_H$, and equations [2.11] must be used to successively compute $\breve{\theta}_{iH}$:

$$\begin{aligned} \breve{\theta}_{1H} &= \arctan\left[\frac{n_3 n_2\left(1-\cos\breve{\Theta}\right)+n_1\sin\breve{\Theta}}{\cos\breve{\Theta}+n_3^2\left(1-\cos\breve{\Theta}\right)}\right]_H \\ \breve{\theta}_{2H} &= -\arcsin\left[n_3 n_1\left(1-\cos\breve{\Theta}\right)-n_2\sin\breve{\Theta}\right]_H \\ \breve{\theta}_{3H} &= \arctan\left[\frac{n_2 n_1\left(1-\cos\breve{\Theta}\right)+n_3\sin\breve{\Theta}}{\cos\breve{\Theta}+n_1^2\left(1-\cos\breve{\Theta}\right)}\right]_H \end{aligned} \qquad [2.20]$$

– *Cylindrical hinge*: $u_i = \breve{u}_{iH}$ must be prescribed; moreover, since the axis of rotation $\bar{\mathbf{n}}$ is given, but the rotation Θ around it is arbitrary, the conditions [2.16] must be enforced, which constitutes two additional constraints for the Tait–Bryan angles:

$$\begin{aligned} \frac{n_1}{n_3} &= \left.\frac{\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2\sin\theta_3+\sin\theta_1\cos\theta_3}{\cos\theta_2\sin\theta_3-\sin\theta_1\sin\theta_2\cos\theta_3+\cos\theta_1\sin\theta_3}\right|_H \\ \frac{n_2}{n_3} &= \left.\frac{\cos\theta_1\sin\theta_2\cos\theta_3+\sin\theta_1\sin\theta_3+\sin\theta_2}{\cos\theta_2\sin\theta_3-\sin\theta_1\sin\theta_2\cos\theta_3+\cos\theta_1\sin\theta_3}\right|_H \end{aligned} \qquad [2.21]$$

REMARK 2.3. When the constraints are applied at interior points of the beam, we have to break the beam in pieces in such a way that the constraints always fall at the boundaries of the subintervals in which the domain has been split. However, while constraints at the end points, according to equation [2.17], restrain the local *absolute* displacements, constraints at the interior points restrain the absolute and/or *relative* displacements between adjacent points. The latter are of the type:

$$\mathbf{u}_H^+ - \mathbf{u}_H^- = \Delta\breve{\mathbf{u}}_H(t), \qquad \mathbf{R}_H^+ - \mathbf{R}_H^- = \Delta\breve{\mathbf{R}}_H(t) \qquad H = A, B \qquad [2.22]$$

i.e. they establish the *jump* in the displacement field at the constrained point and not the displacement itself.

2.1.4 The strain vector

When a beam undergoes a displacement and/or a rotation field, except for special rigid transformations, it changes its shape. This change is called a *deformation*. A measure of the *local* change of shape is said to be a *strain*.

With the aim of defining strain measures, we consider a small neighborhood of the point P and fix our attention on (a) the tangent to the centerline and (b) the attitude of the cross-section before and after the deformation (see Figure 2.4, where

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

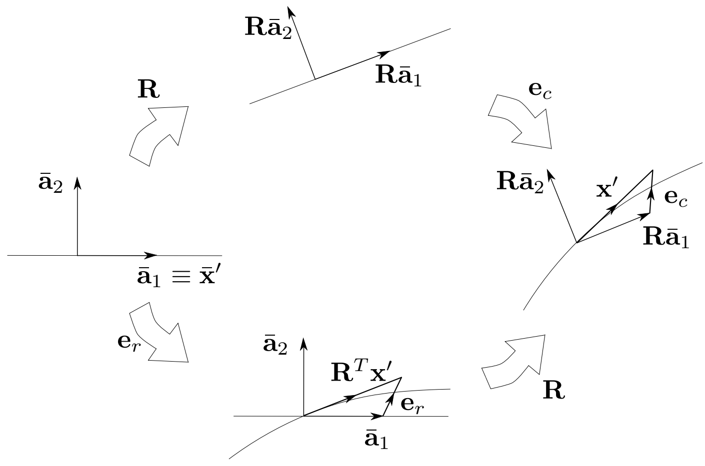


Figure 2.4: Current and reference deformation vectors.

the undeformed and deformed states are represented on the left and on the right sides, respectively; the figure refers to a planar beam, just for clarity of representation). In the reference configuration, the tangent to the centerline is represented by the unit vector $\bar{\mathbf{x}}'$ (where a dash denotes differentiation with respect to s) and the attitude of the section by the normal vector $\bar{\mathbf{a}}_1$; in this state, the two vectors coincide, since $\bar{\mathbf{x}}' \equiv \bar{\mathbf{a}}_1$. In the current configuration, the tangent to the centerline is identified by the vector $\mathbf{x}'$ and the normal to the cross-section by $\mathbf{a}_1 = \mathbf{R}\bar{\mathbf{a}}_1$; the two vectors, however, are no longer aligned, and, moreover, have different length, since $\|\mathbf{x}'\| \neq 1$. Accordingly, the length of $\mathbf{x}'$ accounts for a local *stretch*, and the angle between $\mathbf{x}'$ and $\mathbf{a}_1$ for a local *shear strain*.

Current and reference strains

To define a strain magnitude having a vector character, which describes both stretch and shear, we should first observe that the deformation process includes two contributions: a rotation and a pure deformation. In order to describe the latter, we have to deflate the motion by the rotation. To achieve this goal, we can follow two different processes, illustrated in Figure 2.4, in which *the order of rotation and pure deformation* are exchanged.

In the first process (upper part of the figure), (a) we first apply a rotation $\mathbf{R}$ so that the vector $\bar{\mathbf{a}}_1$ is transformed into $\mathbf{a}_1 = \mathbf{R}\bar{\mathbf{a}}_1$, then (b) we permit $\bar{\mathbf{x}}'$ to stretch and to rotate to match $\mathbf{x}'$. Accordingly, we introduce a strain vector as the difference between

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

the stretched tangent and the rotated unit normal vector, i.e.:

$$\mathbf{e}_c := \mathbf{x}' - \mathbf{R}\bar{\mathbf{a}}_1 \tag{2.23}$$

Since the deformation occurs in the current basis, we will call $\mathbf{e}_c$ *the current strain vector*.

In the second process (bottom part of the figure), (a) we first stretch and rotate $\bar{\mathbf{x}}'$, by transforming it into $\mathbf{R}^T\mathbf{x}'$ (i.e. into the *pulled-back image* of vector $\mathbf{x}'$); then, (b) we apply the rotation $\mathbf{R}$ so that all vectors reach their final position. Accordingly, we introduce a strain vector as the difference between the pulled-back tangent and the (unrotated) unit normal vector, i.e.:

$$\mathbf{e}_r := \mathbf{R}^T\mathbf{x}' - \bar{\mathbf{a}}_1 \tag{2.24}$$

Since the deformation occurs in the reference basis, we will call $\mathbf{e}_r$ *the reference strain vector*.

Other names for $\mathbf{e}_c, \mathbf{e}_r$ are the *left* and *right* strain vectors, respectively, as borrowed from the Cauchy Continuum Mechanics[6]. Both the vectors are differences between the tangent to the centerline and the normal to the section, but they are evaluated *after* or *before* the rotation occurs, respectively. As a result, the two vectors differ from each other, but they are merely related by a rotation, namely:

$$\mathbf{e}_r := \mathbf{R}^T\mathbf{e}_c \tag{2.25}$$

Hence, the reference strain vector is equal to the current one, pulled-back. This property entails that the two strain vectors *have the same components in two different bases*, $\mathbf{e}_c$ in the current basis $\mathcal{B}$, and $\mathbf{e}_r$ in the reference basis $\bar{\mathcal{B}}$:

$$\begin{aligned} \mathbf{e}_c &:= \varepsilon\mathbf{a}_1 + \gamma_2\mathbf{a}_2 + \gamma_3\mathbf{a}_3 \\ \mathbf{e}_r &:= \varepsilon\bar{\mathbf{a}}_1 + \gamma_2\bar{\mathbf{a}}_2 + \gamma_3\bar{\mathbf{a}}_3 \end{aligned} \tag{2.26}$$

as it immediately follows from equations [2.25] and [2.3].[7]

6. The Polar Decomposition Theorem (see [GUR 82]), indeed, states that the deformation gradient $\mathbf{F}$ (relating two material vectors, before and after the deformation, via $d\mathbf{x} = \mathbf{F}\, d\bar{\mathbf{x}}$), can be decomposed in two alternate ways in the product of a rotation tensor $\mathbf{R}$ and a symmetric positive definite *stretch tensor*, $\mathbf{U}$ or $\mathbf{V}$, according to $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$; $\mathbf{U}$ is called the *right stretch tensor* and $\mathbf{V}$ the *left* stretch tensor. Since the transformations must be applied in sequence, from the right to the left, $\mathbf{U}$ is responsible for a stretch that *precedes* the rotation, while $\mathbf{V}$ is a stretch that *follows* the rotation.

7. Therefore $\mathbf{e}_c$, $\mathbf{e}_r$ are vectors *attached to the bases* $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively, as we discussed in the Introduction. Note that, as an exception to the general rule stated there, we did not use the overbar here (i.e. we did not write $\mathbf{e} := \mathbf{e}_c$, $\bar{\mathbf{e}} := \mathbf{e}_r$), since this is suggested by notational convenience.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The scalar components of the strain vector(s) are called the *longitudinal strain* ε and the *transverse strains* γ_2, γ_3. Very often, they are confused with the *unit extension* and the *shear strains*, from the meaning they assume in the linear theory, although this wording should be avoided, in order to not generate confusion.

As a final comment, the current strain appears, in some sense, "more natural" than the reference strain, since it is the strain that an observer attached to the section would see. In contrast, the reference strain calls for introducing "pulled-back vectors", that make its meaning less evident. Nonetheless, the reference strain is consistent with the usual choice made in the Lagrangian approach of elasticity, according to which strains are referred to as the reference configuration. Therefore, *we will adopt the reference vector as a strain measure*; accordingly, we will omit the index r, by letting $\mathbf{e} := \mathbf{e}_r$, when this does not generate ambiguity.

REMARK 2.4. The strain measures (2.4) are *exact*, in the sense that their expressions hold for arbitrarily large displacements and rotations. As a matter of the fact, they vanish when the transformation is a translation ($\mathbf{x}' = \bar{\mathbf{x}}' \equiv \bar{\mathbf{a}}_1,\ \mathbf{R} = \mathbf{I}$) or a rotation ($\mathbf{x}' = \mathbf{R}\bar{\mathbf{a}}_1$).

Physical strain measures

Physical scalar measures for the strains are the *unit extension* and the *shear strains*, which represent, respectively, the relative change of length of an infinitesimal segment of the beam axis, and the change of the (initially right) angles between the beam axis and the two directors lying in the cross-section plane (see Figure 2.5), namely[8] [9]:

$$e := \frac{\|d\mathbf{x}\| - \|d\bar{\mathbf{x}}\|}{\|d\bar{\mathbf{x}}\|} = \|\mathbf{x}'\| - 1, \qquad \alpha_j := \frac{\pi}{2} - \widehat{(\mathbf{x}', \mathbf{a}_j)} \quad j = 2, 3 \qquad [2.27]$$

in which $\|d\bar{\mathbf{x}}\| = ds$ has been accounted for.

From equations [2.24] and [2.26b], it follows that:

$$\mathbf{x}' = \mathbf{R}\left(\bar{\mathbf{a}}_1 + \mathbf{e}\right) = (1 + \varepsilon)\,\mathbf{a}_1 + \gamma_2\mathbf{a}_2 + \gamma_3\mathbf{a}_3 \qquad [2.28]$$

and therefore, equation [2.27] furnishes:

$$e = \sqrt{(1 + \varepsilon)^2 + \gamma_2^2 + \gamma_3^2} - 1 \qquad [2.29]$$

8. The unusual symbols e, α_j here replace the usually adopted symbols ε, γ_j, since the latter assume a different meaning in our context.

9. We found it simpler to perform the analysis in the current configuration. The reader is suggested to redraw the figure by considering "pure deformation" only (i.e. before the rotations occur) and to repeat calculations. Of course, since lengths and angle difference are not affected by the rotation, he/she will find the same results.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

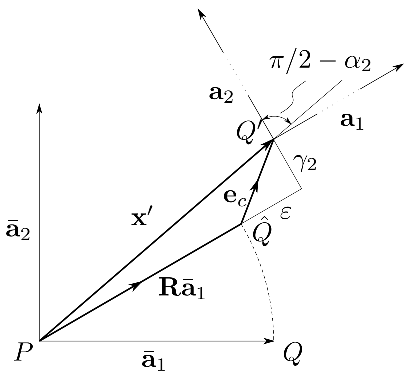


Figure 2.5: Physical strains.

which expresses the unit extension as a function of the longitudinal as well of the transverse strains.

To evaluate α_j, we resort to the definition of scalar product:

$$\cos\left(\frac{\pi}{2}-\alpha_j\right)=\frac{\mathbf{x}'\cdot\mathbf{a}_j}{\|\mathbf{x}'\|}\qquad j=2,3 \tag{2.30}$$

from which we get:

$$\alpha_j=\arcsin\left(\frac{\gamma_j}{\sqrt{(1+\varepsilon)^2+\gamma_2^2+\gamma_3^2}}\right)\qquad j=2,3 \tag{2.31}$$

This expression shows, once again, that the change of the angles depends on all the scalar strains.

When strains ε, γ_j are small of order $\epsilon := \|\mathbf{e}\| \ll 1$, equations [2.29] and [2.31] can be expanded in series to give:

$$\begin{aligned} e &= \varepsilon+\frac{1}{2}\left(\gamma_2^2+\gamma_3^2\right)+\mathrm{O}(\epsilon^3)\\ \alpha_j &= \gamma_j\left(1-\varepsilon\right)+\mathrm{O}(\epsilon^3)\end{aligned} \tag{2.32}$$

Therefore, when $\epsilon\to 0$, then $e\to\varepsilon$ $\alpha_j\to\gamma_j$, so that the unit extension can be confused with the longitudinal strain and the transverse strain with the shear strain.

2.1.5 The curvature vector

The strain vector, discussed before, is not sufficient, by itself, to describe the local change of shape of the body, since the beam could undergo no stretch and no shear,

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

but it could bend or twist itself. Therefore, we have to define another quantity able to account for the *curvature* of the beam. To introduce the concept in a very simple way, we start from the elementary notion of *curvature of a line*, which is well-known from differential geometry of curves. Then, we will adapt this definition to our mechanical problem.

The curvature of a line and the Frenet formulas

Let us consider a curve $\mathcal{S}$ (Figure 2.6(a), on which a curvilinear abscissa s has been introduced, and let $\mathcal{B}_f := (\mathbf{a}_t(s), \mathbf{a}_n(s), \mathbf{a}_b(s))$ a right orthogonal triad, called *TNB* or Frenet triad[10], depending on s, in which: $\mathbf{a}_t$ is unit *tangent* vector, $\mathbf{a}_n$ is the unit *normal* vector (spanning, with $\mathbf{a}_t$, the *osculating plane* at s) and $\mathbf{a}_b := \mathbf{a}_t \times \mathbf{a}_n$ is the unit *binormal* vector completing the triad. The derivatives of the unit vectors are expressed by the *Frenet formulas*:

$$\mathbf{a}_t' = k\mathbf{a}_n, \qquad \mathbf{a}_b' = -\tau\mathbf{a}_n, \qquad \mathbf{a}_n' = \tau\mathbf{a}_b - k\mathbf{a}_t \tag{2.33}$$

where k is the *curvature* and τ is the *torsion* of the curve at s. When $k > 0$, $\mathbf{a}_n$ points toward the concavity of the curve. Equations [2.33a,b] give us an interpretation of the curvature (and torsion) as the “velocity” by which the tangent (or the binormal) to the curve rotates in traveling the curve itself with uniform speed; the faster the tangent (or binormal) rotates, the larger the curvature (or torsion)[11].

The Frenet formulas can be more conveniently recast in tensor form as:

$$\mathbf{a}_\alpha' = \mathbf{K}_f\mathbf{a}_\alpha, \qquad \alpha = t, n, b \tag{2.34}$$

in which $\mathbf{K}_f$ is the *Frenet curvature tensor*, which transforms the unit vectors of the Frenet triad in their derivatives[12]. The components of $\mathbf{K}_f$ in $\mathcal{B}_f$ are $\mathbf{a}_\alpha \cdot \mathbf{K}_f\mathbf{a}_\beta = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta'$; since, from equations [2.33], we have:

$$\begin{aligned} &\mathbf{a}_t' \cdot \mathbf{a}_t = \mathbf{a}_n' \cdot \mathbf{a}_n = \mathbf{a}_b' \cdot \mathbf{a}_b = 0, \quad && \mathbf{a}_t' \cdot \mathbf{a}_b = -\mathbf{a}_b' \cdot \mathbf{a}_t = 0 \\ &\mathbf{a}_t' \cdot \mathbf{a}_n = -\mathbf{a}_n' \cdot \mathbf{a}_t = k, && \mathbf{a}_n' \cdot \mathbf{a}_b = -\mathbf{a}_b' \cdot \mathbf{a}_n = \tau \end{aligned} \tag{2.35}$$

10. Commonly, TNB stands for tangent–normal–binormal, which is an alternative way to indicate the Frenet triad.

11. Formula [2.33a] can be easily justified on a geometrical ground, as shown in (Figure 2.6(b). Since $\mathbf{a}_t(s + ds) - \mathbf{a}_t(s) = d\theta\mathbf{a}_n + O(ds^2)$, where $d\theta$ is the angle between the two vectors, then $\mathbf{a}_t' = (d\theta/ds)\mathbf{a}_n = k\mathbf{a}_n$, where $k := 1/R(s)$ is the inverse of the radius of the osculating circle. Analogously, the second formula defines torsion as a measure of how the curve deviates from the osculating plane; if the curve is locally planar, then the torsion vanishes. The third equation follows from the identity $\mathbf{a}_n' = (\mathbf{a}_b \times \mathbf{a}_t)'$, when equations [2.33a,b] are used.

12. Therefore, $\mathbf{K}_f$ transforms *any* vector $\mathbf{w}$, of constant modulus and attached to the basis, in its derivative $\mathbf{w}'$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

then $\mathbf{K}_f$ is skew-symmetric, and, moreover, its scalar representation on $\mathcal{B}_f$ is:

$$\boldsymbol{K}_f := [\mathbf{K}_f]_{\mathcal{B}_f} = \begin{bmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \qquad [2.36]$$

The skew-symmetric tensor admits $\mathbf{k}_f := k\mathbf{a}_b + \tau\mathbf{a}_t$ as the *axial vector*, for which the following property holds:

$$\mathbf{a}'_\alpha = \mathbf{k}_f \times \mathbf{a}_\alpha, \qquad \alpha = t, n, b \qquad [2.37]$$

REMARK 2.5. The Frenet axial vector $\mathbf{k}_f$ (as well the matrix curvature $\boldsymbol{K}_f$) possesses only two non-zero components along the binormal and the tangent to the curve. This is due to the special character of the Frenet triad, in which the normal and binormal are TNB directions. If we, in contrast, choose two generic directions in the normal plane, then, of course, all the components of the axial vector are non-zero.

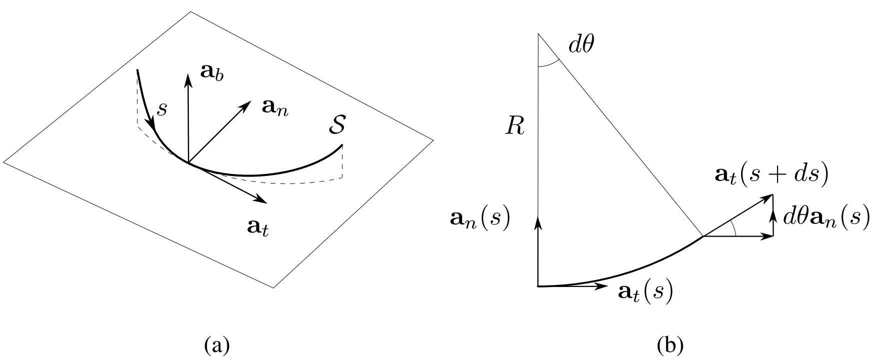


Figure 2.6: Frenet triad and formulas: (a) intrinsic triad, (b) unit vector derivatives.

The curvature of the beam

In defining the curvature of a beam, we will closely follow the definition of the curvature of a linefernet! principal curvature and torsion of a line. However, two main differences must be taken into account with respect to that case. First, (a) we are *not* interested in determining how fast the intrinsic basis $\mathcal{B}_f$, attached to the curve, rotates in traveling the centerline, but, in contrast, we are interested in evaluating *how fast the basis* $\mathcal{B}$ *of the directors, attached to the cross-sections, rotates*. That will be a measure of the *change of attitude* of the cross-section along the centerline. Second, (b) while the Frenet curvature refers to the current arc length, it seems more suitable, in defining

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

the mechanical curvature, to refer to the unstretched arc length s as a simple example shows[13].

Guided by the previous considerations, we define the *current curvature tensor* $\mathbf{K}_c$ as the linear operator which transforms $\mathbf{a}_j$ into its derivative $\mathbf{a}_j'$, i.e.:

$$\mathbf{a}_j' = \mathbf{K}_c \mathbf{a}_j \qquad j = 1, 2, 3 \tag{2.38}$$

To express $\mathbf{K}_c$ in terms of the rotation tensor, we differentiate equation [2.3] with respect to s, to get:

$$\mathbf{a}_j' = \mathbf{R}' \bar{\mathbf{a}}_j = \mathbf{R}' \mathbf{R}^T \mathbf{a}_j \tag{2.39}$$

in which we accounted for $\bar{\mathbf{a}}_j' = 0$ (since the beam is initially straight) and, moreover, we again used equation [2.3]. By comparing equations [2.38] and [2.39], we find:

$$\mathbf{K}_c = \mathbf{R}' \mathbf{R}^T \tag{2.40}$$

Since, by differentiating equation [2.4], we have $\left(\mathbf{R}\mathbf{R}^T\right)' = \mathbf{R}'\mathbf{R}^T + \mathbf{R}\mathbf{R}'^T = \mathbf{0}$, then $\mathbf{K}_c = -\mathbf{K}_c^T$, i.e. *the current curvature tensor is skew-symmetric*. Hence, equation [2.38] is equivalent to:

$$\mathbf{a}_j' = \mathbf{k}_c \times \mathbf{a}_j \qquad j = 1, 2, 3 \tag{2.41}$$

where the *current curvature vector* $\mathbf{k}_c$ is the axial vector of $\mathbf{K}_c$.

Similar to what we did for strain vectors, we can also define a *reference curvature tensor* $\mathbf{K}_r$, which performs the same operation of $\mathbf{K}_c$, but *before the rotation* $\mathbf{R}$ *occurred*, i.e. working on the pulled-back images of $\mathbf{a}_j$ and $\mathbf{a}_j'$. Accordingly, we let:

$$\mathbf{R}^T \mathbf{a}_j' = \mathbf{K}_r \bar{\mathbf{a}}_j \tag{2.42}$$

To obtain an expression for $\mathbf{K}_r$ in terms of the rotation tensor, we substitute $\mathbf{a}_j' = \mathbf{R}' \bar{\mathbf{a}}_j$ into equation [2.42] and we get:

$$\mathbf{K}_r = \mathbf{R}^T \mathbf{R}' \tag{2.43}$$

---

13. Let us discuss the example of a planar beam, whose cross-sections remain orthogonal to the axis. If we bend the beam in a circular shape, without stretching its axis, we observe that the mechanical curvature of the beam coincides with the geometrical curvature (i.e. the inverse of the circle radius). If, however, we superimpose a displacement field in which all points move along the local external normal to the circle, the shape of the beam is still circular, but of larger radius. The geometrical curvature, of course, decreases in this transformation, but we say, on a physical ground, that the beam undergoes a pure stretching, not a bending. Indeed, if we evaluate the derivatives of the unit vectors of the triad with respect to the unstretched abscissa, the curvature does not change in the second transformation.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Since $\left(\mathbf{R}^T\mathbf{R}\right)' = \mathbf{R}^T\mathbf{R}' + \mathbf{R}'^T\mathbf{R} = \mathbf{0}$, then $\mathbf{K}_r = -\mathbf{K}_r^T$, i.e. *the reference curvature tensor is skew-symmetric*. Consequently, a *reference curvature vector* $\mathbf{k}_r$ exists, such that:

$$\mathbf{R}^T\mathbf{a}_j' = \mathbf{k}_r \times \bar{\mathbf{a}}_j \qquad j = 1, 2, 3 \qquad [2.44]$$

The two tensors, $\mathbf{K}_c$ and $\mathbf{K}_r$, differ from each other, but they are related by:

$$\mathbf{K}_r = \mathbf{R}^T\mathbf{K}_c\mathbf{R} \qquad [2.45]$$

this showing that they *have the same components in the two bases*, namely the current one in the current basis $\mathcal{B}$ and the reference one in the reference basis $\bar{\mathcal{B}}$ [14, 15]; since they are antisymmetric, their scalar representation is:

$$\boldsymbol{K} := [\mathbf{K}_c]_{\mathcal{B}} \equiv [\mathbf{K}_r]_{\bar{\mathcal{B}}} = \begin{pmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{pmatrix} \qquad [2.46]$$

The same property, of course, holds for the relevant axial vectors. Therefore:

$$\begin{aligned} \mathbf{k}_c &:= \kappa_1\mathbf{a}_1 + \kappa_2\mathbf{a}_2 + \kappa_3\mathbf{a}_3 \\ \mathbf{k}_r &:= \kappa_1\bar{\mathbf{a}}_1 + \kappa_2\bar{\mathbf{a}}_2 + \kappa_3\bar{\mathbf{a}}_3 \end{aligned} \qquad [2.47]$$

where κ_1 is called the *torsional curvature* and κ_2, κ_3 are the *bending curvatures*. The axial vectors are therefore related, as the strain vectors are (equation [2.25]), namely:

$$\mathbf{k}_c = \mathbf{R}\mathbf{k}_r \qquad [2.48]$$

Although the definition of the current curvature appears to be more natural, according to previous considerations, *we assume the reference curvature as a measure of strain*. Accordingly, we will omit the index r, by letting $\mathbf{K} := \mathbf{K}_r$ and $\mathbf{k} := \mathbf{k}_r$, when this does not generate ambiguity.

REMARK 2.6. Equations [2.41] permit us to evaluate the space-derivative of a vector attached to the basis $\mathcal{B}$, e.g. $\mathbf{w} = \sum_{i=1}^{3} w_i\mathbf{a}_i$. Since $\mathbf{w}' = \sum_{i=1}^{3}(w_i'\mathbf{a}_i + w_i\mathbf{a}_i')$, it follows that:

$$\mathbf{w}' = \sum_{i=1}^{3} w_i'\mathbf{a}_i + \mathbf{k}_c \times \mathbf{w} \qquad [2.49]$$

which is known as *Poisson formula.*

14. As a matter of fact,
$mathbfa_i \cdot \mathbf{K}_c\mathbf{a}_j = \mathbf{a}_i \cdot \mathbf{R}'\mathbf{R}^T\mathbf{a}_j = \mathbf{R}\bar{\mathbf{a}}_i \cdot \mathbf{R}'\mathbf{R}^T\mathbf{R}\bar{\mathbf{a}}_j = \bar{\mathbf{a}}_i \cdot \mathbf{R}^T\mathbf{R}'\bar{\mathbf{a}}_j = \bar{\mathbf{a}}_i \cdot \mathbf{K}_r\bar{\mathbf{a}}_j$, having used equation [2.4] and the identity $\mathbf{L}\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{L}^T\mathbf{v}$.

15. Therefore $\mathbf{K}_c, \mathbf{K}_r$ are tensors *attached to the bases* $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively, as we discussed in the Introduction. As an exception to the general rule we established there, we avoided using the overbar here (i.e. we did not write $\mathbf{K} := \mathbf{K}_c$, $\bar{\mathbf{K}} := \mathbf{K}_r$), as suggested by notational convenience and to avoid conflicts with other symbols. Analogous comments hold for the axial vectors, to be introduced in the equation [2.47].

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

REMARK 2.7. The scalar representation $[\mathbf{K}_c]_{\mathcal{B}}$ can, of course, be obtained via direct evaluation of the components of $\mathbf{R}'\mathbf{R}^T$ in $\mathcal{B}$, instead of resorting to the components of $\mathbf{R}^T\mathbf{R}'$ in $\bar{\mathcal{B}}$. The calculation, however, is much more difficult, since the basis $\mathcal{B}$ depends on s, and therefore $[\mathbf{R}']_{\mathcal{B}} \neq [\mathbf{R}]'_{\mathcal{B}}$, i.e. the matrix of derivatives is *not* the derivative of the matrix. The reader interested in the question can adapt formula [3.13] to the problem at hand, to prove that $[\mathbf{R}']_{\mathcal{B}} = [\mathbf{R}]'_{\mathcal{B}} + [\mathbf{K}_c]_{\mathcal{B}}[\mathbf{R}]_{\mathcal{B}} - [\mathbf{R}]_{\mathcal{B}}[\mathbf{K}_c]_{\mathcal{B}}$. Then, by post-multiplying this equation by $[\mathbf{R}]^T_{\mathcal{B}}$, taking into account that $[\mathbf{K}_c]_{\mathcal{B}} := [\mathbf{R}']_{\mathcal{B}}[\mathbf{R}]^T_{\mathcal{B}}$ and simplifying, it follows that $[\mathbf{K}_c]_{\mathcal{B}} = [\mathbf{R}]^T_{\mathcal{B}}[\mathbf{R}]'_{\mathcal{B}}$; since $[\mathbf{R}]_{\mathcal{B}} = [\mathbf{R}]_{\bar{\mathcal{B}}}$, then $[\mathbf{K}_c]_{\mathcal{B}} = [\mathbf{R}]^T_{\bar{\mathcal{B}}}[\mathbf{R}]'_{\bar{\mathcal{B}}}$.

2.1.6 The strain–displacement relationships

The equations relating strains and displacements are named *strain-displacement relationships*. To obtain them, we have to express the strain vector $\mathbf{e}$ and the curvature vector $\mathbf{k}$ in terms of the translation $\mathbf{u}$ and the rotation $\mathbf{R}$.

By using $\mathbf{x}' = \bar{\mathbf{a}}_1 + \mathbf{u}'$ in equation [2.24], we have:

$$\mathbf{e} = \mathbf{R}^T(\bar{\mathbf{a}}_1 + \mathbf{u}') - \bar{\mathbf{a}}_1 \tag{2.50}$$

whose scalar representation in $\bar{\mathcal{B}}$, in matrix form, is:

$$\boldsymbol{e} = \boldsymbol{R}^T(\bar{\boldsymbol{a}}_1 + \boldsymbol{u}') - \bar{\boldsymbol{a}}_1 \tag{2.51}$$

or, in extended form:

$$\begin{pmatrix}\varepsilon\\ \gamma_2\\ \gamma_3\end{pmatrix} = \begin{bmatrix} \cos\theta_2\cos\theta_3 & \cos\theta_2\sin\theta_3 & -\sin\theta_2\\ \sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 & \sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\cos\theta_2\\ \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 & \cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\cos\theta_2 \end{bmatrix} \begin{pmatrix}1+u_1'\\ u_2'\\ u_3'\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{2.52}$$

where equations [2.26b], [2.2] and [2.6] have been accounted for.

To express the scalar components in $\bar{\mathcal{B}}$ of the curvature in terms of rotations, $\boldsymbol{K} = \boldsymbol{R}^T\boldsymbol{R}'$ is used, according to equation [2.43], in which $\boldsymbol{R}'$ is evaluated by differentiating, term-by-term, matrix $\boldsymbol{R}$. Then, the components of the axial vector are extracted by the matrix, according to equation [2.46]. After simplification, the following, quite simple, relations are found:

$$\begin{pmatrix}\kappa_1\\ \kappa_2\\ \kappa_3\end{pmatrix} = \begin{pmatrix}1 & 0 & -\sin\theta_2\\ 0 & \cos\theta_1 & \sin\theta_1\cos\theta_2\\ 0 & -\sin\theta_1 & \cos\theta_1\cos\theta_2\end{pmatrix}\begin{pmatrix}\theta_1'\\ \theta_2'\\ \theta_3'\end{pmatrix} \tag{2.53}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

or in compact form[16]:

$$\boldsymbol{k} = \boldsymbol{B}_\omega \boldsymbol{\theta}' \qquad [2.54]$$

Equations [2.52] and [2.53] are the strain-displacement relationships sought for. They show that for differentiability, displacements and rotations must be continuous along the centerline.

REMARK 2.8. While the column matrix $\boldsymbol{k}$ is the collection of the components of a vector, $\boldsymbol{\theta}$ is *not* the collection of the components of a vector. Therefore, occasionally, it is called a *pseudo-vector* (by Argyris [ARG 82]).

2.1.7 The velocity and spin fields

Displacement and rotation fields are not sufficient to describe the state of the body, but additional quantities, having the meaning of time-rates, must be introduced. Here we define velocity and spin.

Velocity field

By taking the first time-derivative (denoted by a dot) of the position vector $\mathbf{x}(s,t)$ and using equation [2.1], we obtain the *velocity field*:

$$\mathbf{v} := \dot{\mathbf{x}}(s,t) \equiv \dot{\mathbf{u}}(s,t) \qquad [2.55]$$

which, due to equation [2.2], entails that:

$$\mathbf{v} := \dot{u}_1(s,t)\bar{\mathbf{a}}_1 + \dot{u}_2(s,t)\bar{\mathbf{a}}_2 + \dot{u}_3(s,t)\bar{\mathbf{a}}_3 \qquad [2.56]$$

Spin field

By exploiting the analogy with the space-derivatives of the unit vectors (equation [2.38]), we introduce a *spin tensor* $\mathbf{W}$ as the linear operator that transforms $\mathbf{a}_j$ into its time-derivative $\dot{\mathbf{a}}_j$, i.e.:

$$\dot{\mathbf{a}}_j = \mathbf{W}\mathbf{a}_j \qquad j = 1,2,3 \qquad [2.57]$$

16. The matrix $\boldsymbol{B}_\omega$ possesses columns of unitary modulus, i.e. it represents a basis of unit vectors. We will investigate its meaning (and understand the reason of the index ω) in the next section.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By time-differentiating equation [2.3] and using the same equation, we get $\dot{\mathbf{a}}_j = \dot{\mathbf{R}}\bar{\mathbf{a}}_j = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{a}_j$, from which:

$$\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T \qquad [2.58]$$

Since $\left(\mathbf{R}\mathbf{R}^T\right)^{\cdot} = \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \mathbf{0}$, then $\mathbf{W} = -\mathbf{W}^T$, i.e. *the spin tensor is skew-symmetric*. Therefore, equation [2.57] can also be written as:

$$\dot{\mathbf{a}}_j = \boldsymbol{\omega} \times \mathbf{a}_j \qquad j = 1, 2, 3 \qquad [2.59]$$

where the *spin* (or angular velocity) *vector* $\boldsymbol{\omega}$ is the axial vector of $\mathbf{W}$.

By using arguments similar to those used for the curvature tensor, we get the scalar representations *in the current basis* for the spin tensor and vector, namely, $\boldsymbol{W} := [\mathbf{W}]_{\mathcal{B}} = \boldsymbol{R}^T\dot{\boldsymbol{R}}$, or:

$$\boldsymbol{W} := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad [2.60]$$

and:

$$\boldsymbol{\omega} := \omega_1\mathbf{a}_1 + \omega_2\mathbf{a}_2 + \omega_3\mathbf{a}_3 \qquad [2.61]$$

Finally, in analogy with equations [2.53], we find the components of $\boldsymbol{\omega}$ in $\mathcal{B}$:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\sin\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1\cos\theta_2 \\ 0 & -\sin\theta_1 & \cos\theta_1\cos\theta_2 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} \qquad [2.62]$$

or in compact form:

$$\boldsymbol{\omega} = \boldsymbol{B}_\omega\dot{\boldsymbol{\theta}} \qquad [2.63]$$

REMARK 2.9. By bearing in mind the analogy with the curvature tensors, the spin tensor and vector defined above should be meant as the *current spin tensor* $\mathbf{W} \equiv \mathbf{W}_c$ *and vetor* $\boldsymbol{\omega} \equiv \boldsymbol{\omega}_c$. Of course, a *reference spin tensor* could also be defined as $\mathbf{W}_r := \mathbf{R}^T\dot{\mathbf{R}}$, with axial vector $\boldsymbol{\omega}_r$, such that $\mathbf{R}^T\dot{\mathbf{a}}_j = \boldsymbol{\omega}_r \times \bar{\mathbf{a}}_j$; consequently, $\boldsymbol{\omega}_c = \mathbf{R}\boldsymbol{\omega}_r$. However, different from the strains, $\mathbf{W}_r$ is not useful to our treatment, as will be clearer later. This is due to the fact that strains refer to the reference configuration, while virtual motions and inertia forces (which the spin contributes to) are referred to the current configuration.

REMARK 2.10. Equations [2.59] permit us to evaluate the time-derivative of a vector attached to the basis $\mathcal{B}$, e.g. $\mathbf{w} = \sum_{i=1}^{3} w_i\mathbf{a}_i$. Since $\dot{\mathbf{w}} = \sum_{i=1}^{3}(\dot{w}_i\mathbf{a}_i + w_i\dot{\mathbf{a}}_i)$, it follows that (*Poisson formula*):

$$\dot{\mathbf{w}} = \sum_{i=1}^{3} \dot{w}_i\mathbf{a}_i + \boldsymbol{\omega} \times \mathbf{w} \qquad [2.64]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

REMARK 2.11. The Evaluation of the second time-derivative of $\mathbf{w} = \sum_{i=1}^{3} w_i \mathbf{a}_i$ could be needed. It happens, for example, when the acceleration of a point is desired in local (rotating) coordinates. By differentiating the previous equation and using equation [2.64] again, we obtain:

$$\ddot{\mathbf{w}} = \sum_{i=1}^{3} \ddot{w}_i \mathbf{a}_i + 2\boldsymbol{\omega} \times \sum_{i=1}^{3} \dot{w}_i \mathbf{a}_i + \dot{\boldsymbol{\omega}} \times \mathbf{w} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{w}) \qquad [2.65]$$

The Fundamental Formula of Rigid Kinematics

We will resort, occasionally, to a 3D-model of beams with rigid cross-sections, to identify kinetic or elastic quantities for the 1D-model. In view of these developments, we need to express the velocity field of the rigid cross-section. To this end, let us consider the generic cross-section and denote its centroid by G[17]. In the reference configuration, the position of a point Q on the section is $\bar{\mathbf{x}}_Q = \bar{\mathbf{x}}_G + \bar{\mathbf{r}}$, where $\bar{\mathbf{r}} := r_2 \bar{\mathbf{a}}_2 + r_3 \bar{\mathbf{a}}_3$[18]. In the current configuration, the position of Q is:

$$\mathbf{x}_Q = \mathbf{x}_G (s,t) + \mathbf{R}(s,t)\, \bar{\mathbf{r}} \qquad [2.66]$$

By differentiating the previous equations with respect to the time, and accounting for the independence of time of $\bar{\mathbf{r}}$, it follows:

$$\begin{aligned} \mathbf{v}_Q &= \mathbf{v}_G (s,t) + \dot{\mathbf{R}}(s,t)\, \bar{\mathbf{r}} \\ &= \mathbf{v}_G (s,t) + \dot{\mathbf{R}}(s,t)\, \mathbf{R}^T(s,t)\, \mathbf{r} \end{aligned} \qquad [2.67]$$

where $\mathbf{v}_Q := \dot{\mathbf{x}}_Q$, $\mathbf{v}_G := \dot{\mathbf{x}}_G$ and $\mathbf{r} = r_2 \mathbf{a}_2 + r_3 \mathbf{a}_3$ is the transformed vector $\bar{\mathbf{r}}$. By accounting for equation [2.58] and omitting the arguments, we finally get:

$$\mathbf{v}_Q = \mathbf{v}_G + \mathbf{W}\mathbf{r} \qquad [2.68]$$

or:

$$\mathbf{v}_Q = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r} \qquad [2.69]$$

This is known as the "Fundamental Formula of Rigid Kinematics".

17. We call P a point of the 1D-model (although it could be thought as the centroid of the underlying 3D-model), but we use G for the centroid of the cross-section of the 3D-model and Q for a generic point which lies on the cross-section.

18. We avoid the overbar on the coordinates r_i, since we will always need to express $\bar{\mathbf{r}}$ in $\bar{\mathcal{B}}$ and $\mathbf{r} = \mathbf{R}\bar{\mathbf{r}}$ in $\mathcal{B}$, which have the same components r_i.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

A geometrical interpretation of the spin vector: the spin basis

We now want to give a geometrical interpretation of the spin vector $\boldsymbol{\omega}$, which will be useful for further developments. We saw that a finite rotation is a composition of three elementary rotations around three axes, namely $(\bar{\mathbf{a}}_3, \theta_3)$,$(\check{\mathbf{a}}_2, \theta_2)$ and $(\tilde{\mathbf{a}}_1, \theta_1)$, in the order. If we perturb the elementary rotations, i.e. if we consider the composition of $(\bar{\mathbf{a}}_3, \theta_3 + \delta\theta_3)$,$(\check{\mathbf{a}}_2, \theta_2 + \delta\theta_2)$ and $(\tilde{\mathbf{a}}_1, \theta_1 + \delta\theta_1)$, and then linearize in the perturbations, we obtain a motion which differs from the original one by a *sum* of three infinitesimal rotations $(\bar{\mathbf{a}}_3, \delta\theta_3)$,$(\check{\mathbf{a}}_2, \delta\theta_2)$ and $(\tilde{\mathbf{a}}_1, \delta\theta_1)$. By considering an infinitesimal interval of time δt and denoting by $\dot{\theta}_i := \delta\theta_i/\delta t$ the time-derivatives of the elementary rotations, we obtain the spin vector as a sum of three elementary spin vectors:

$$\boldsymbol{\omega} = \dot{\theta}_1\tilde{\mathbf{a}}_1 + \dot{\theta}_2\check{\mathbf{a}}_2 + \dot{\theta}_3\bar{\mathbf{a}}_3 \qquad [2.70]$$

It should be stressed that *the basis* $\mathcal{B}_\omega := (\tilde{\mathbf{a}}_1, \check{\mathbf{a}}_2, \bar{\mathbf{a}}_3)$ *is non-orthogonal*; we will refer to it as the *spin basis*.

Of course, we can represent the vector $\boldsymbol{\omega}$ in the two (orthogonal) bases $\bar{\mathcal{B}}$ or $\mathcal{B}$, by letting $\boldsymbol{\omega} = \sum_{i=1}^{3} \bar{\omega}_i \bar{\mathbf{a}}_i$ or $\boldsymbol{\omega} = \sum_{i=1}^{3} \omega_i \mathbf{a}_i$: in matrix form, $\bar{\boldsymbol{\omega}} := (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)^T$ or $\boldsymbol{\omega} := (\omega_1, \omega_2, \omega_3)^T$. Consequently, we have:

$$\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{B}}_\omega\dot{\boldsymbol{\theta}}, \qquad \boldsymbol{\omega} = \boldsymbol{B}_\omega\dot{\boldsymbol{\theta}} \qquad [2.71]$$

Here, the matrices $\bar{\boldsymbol{B}}_\omega$ and $\boldsymbol{B}_\omega$ collect, column-wise, the director cosines of $(\tilde{\mathbf{a}}_1, \check{\mathbf{a}}_2, \bar{\mathbf{a}}_3)$ in the reference or current bases, respectively; we will call them the *spin-basis matrices*. To build them up, we start from the reference basis, in which, with the help of Figure 2.3, we have:

$$\begin{aligned} \check{\mathbf{a}}_2 &= -\sin\theta_3\bar{\mathbf{a}}_1 + \cos\theta_3\bar{\mathbf{a}}_2 \\ \tilde{\mathbf{a}}_1 &= \cos\theta_2\check{\mathbf{a}}_1 - \sin\theta_2\bar{\mathbf{a}}_3 \\ &= \cos\theta_2\left(\cos\theta_3\bar{\mathbf{a}}_1 + \sin\theta_3\bar{\mathbf{a}}_2\right) - \sin\theta_2\bar{\mathbf{a}}_3 \end{aligned} \qquad [2.72]$$

from which we get:

$$\bar{\boldsymbol{B}}_\omega := \begin{pmatrix} \cos\theta_2\cos\theta_3 & -\sin\theta_3 & 0 \\ \cos\theta_2\sin\theta_3 & \cos\theta_3 & 0 \\ -\sin\theta_2 & 0 & 1 \end{pmatrix} \qquad [2.73]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The same procedure could be applied in the current basis to find $\boldsymbol{B}_\omega$; alternatively, since $\boldsymbol{\omega} = \boldsymbol{R}^T\bar{\boldsymbol{\omega}} = \boldsymbol{R}^T\bar{\boldsymbol{B}}_\omega\dot{\boldsymbol{\theta}}$[19], then $\boldsymbol{B}_\omega = \boldsymbol{R}^T\bar{\boldsymbol{B}}_\omega$ and therefore[20]:

$$\boldsymbol{B}_\omega = \begin{pmatrix} 1 & 0 & -\sin\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1\cos\theta_2 \\ 0 & -\sin\theta_1 & \cos\theta_1\cos\theta_2 \end{pmatrix} \qquad [2.74]$$

which is just the matrix appearing in equation [2.62] (and, for the analogy between the curvature vector and the spin vector, also in equation [2.53]).

2.1.8 The velocity gradients and strain-rates

In this section, we introduce two new vector quantities, the velocity and spin gradients. These are a measure of *how displacement time-gradients vary in space*. After that, we will link them to the strain-rates, which are a measure of *how displacement space-gradients vary in time*.

The velocity and the spin gradients

Let us consider two material body-points P, Q on the beam axis, infinitely close to each other, whose relative position, in the reference configuration, is $\Delta\bar{\mathbf{x}} := \bar{\mathbf{x}}_Q - \bar{\mathbf{x}}_P$ (Figure 2.7). In the current, configuration, their relative position changes into $\Delta\mathbf{x} := \mathbf{x}_Q - \mathbf{x}_P$, and the two points have velocities $\mathbf{v}_P$ and $\mathbf{v}_Q$, respectively. We define:

$$\mathbf{g} := \lim_{\|\Delta\bar{\mathbf{x}}\|\to 0} \frac{\mathbf{v}_Q - \mathbf{v}_P}{\|\Delta\bar{\mathbf{x}}\|} = \frac{\partial\mathbf{v}}{\partial s} \equiv \mathbf{v}' \qquad [2.75]$$

and call it the *velocity gradient*. Note that the incremental ratio is evaluated with respect to the undeformed length $\|\Delta\bar{\mathbf{x}}\|$, *not* to the current $\|\Delta\mathbf{x}\|$. For this reason, the vector $\mathbf{g}$ is a *material* gradient, which measures the variation of velocity in passing from a point to a closer one, compared with the distance they had in the reference configuration. This gradient, however, is *not* a measure of the strain-rate, since, in a locally rigid motion, as that described by equation [2.69], the same definition furnishes:

$$\mathbf{w} := \lim_{\|\Delta\bar{\mathbf{x}}\|\to 0} \frac{\boldsymbol{\omega}\times\Delta\mathbf{x}}{\|\Delta\bar{\mathbf{x}}\|} = \boldsymbol{\omega}\times\mathbf{x}' \qquad [2.76]$$

(to be referred to as the *rigid velocity gradient*), and therefore it does not vanish. However, we can express the velocity gradient $\mathbf{g}$ as the sum of two vectors, one

19. Indeed, if the representation of a vector $\mathbf{w}$ is known in $\mathcal{B}$, i.e. $\mathbf{w} = \sum_{j=1}^3 w_j\mathbf{a}_j$, then $\bar{w}_i := \bar{\mathbf{a}}_i\cdot\mathbf{w} = \sum_{j=1}^3 \bar{\mathbf{a}}_i\cdot\mathbf{R}\bar{\mathbf{a}}_j w_j = R_{ij}w_j$; therefore $\bar{\boldsymbol{w}} = \boldsymbol{R}\boldsymbol{w}$.

20. Note that $\boldsymbol{B}_\omega$, $\bar{\boldsymbol{B}}_\omega$ do *not* transform as a tensor, since $\boldsymbol{\theta}$ is not a vector.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

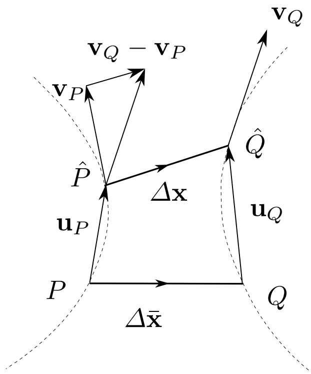


Figure 2.7: Velocity gradient.

accounting for the rigid component of motion, $\mathbf{w}$, the other for the pure stretching part, $\mathbf{d}$, by letting $\mathbf{g} = \mathbf{w} + \mathbf{d}$, from which we get[21]:

$$\mathbf{d} = \mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}' \tag{2.77}$$

to be referred to as the *stretching velocity gradient*. It represents the gradient of velocity in a pure deformation, since it vanishes in a rigid motion.

Similar to what was done for the velocity field $\mathbf{v}$, we can define a *spin gradient*:

$$\mathbf{s} := \lim_{\|\Delta\bar{\mathbf{x}}\| \to 0} \frac{\boldsymbol{\omega}_Q - \boldsymbol{\omega}_P}{\|\Delta\bar{\mathbf{x}}\|} = \frac{\partial \boldsymbol{\omega}}{\partial s} \equiv \boldsymbol{\omega}' \tag{2.78}$$

Since in a locally rigid motion $\boldsymbol{\omega}_Q = \boldsymbol{\omega}_P$, the spin gradient is also a measure of stretching, i.e. no deflation is needed.

Strain-rates versus velocity gradients

We define *strain-rate* and *curvature-rate* as the time-derivatives of the strain and curvature vectors, namely $\dot{\mathbf{e}}$ and $\dot{\mathbf{k}}$, respectively. They are a measure of how a strain at

---

21. This additive decomposition is similar to that performed in the Cauchy Continuum Mechanics, usually denoted as $\mathbf{G} = \mathbf{W} + \mathbf{D}$ [GUR 82], in which $\mathbf{G}$ is the velocity gradient tensor, $\mathbf{W}$ is the spin tensor and $\mathbf{D}$ is the *velocity of deformation tensor*. However, different from our approach, all these are *spatial* gradients, i.e. expressed in terms of current coordinates. Moreover, they are tensors and not vectors, since they refer to a 3D-continuum.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

a fixed point s varies in time. On the other hand, we observe that the stretching velocity gradient $\mathbf{d}$ and the spin gradient $\mathbf{s}$, which has just now been introduced, are a measure of how the stretching velocity at a given time t varies in space. Therefore, both pairs of quantities concern deformation (i.e. space-gradient) and velocity (i.e. time-gradient), although the operation of space- and time-differentiation are exchanged. One could ask himself if the two pairs of quantities are somewhat related.

To answer the question, we start from the definition $\mathbf{d} := \mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}'$ of the stretching velocity gradient, and rearrange it as follows:

$$\mathbf{d} = \dot{\mathbf{x}}' - \dot{\mathbf{R}}\mathbf{R}^T\mathbf{x}' = \dot{\mathbf{x}}' + \mathbf{R}\dot{\mathbf{R}}^T\mathbf{x}' = \mathbf{R}\left(\mathbf{R}^T\dot{\mathbf{x}}' + \dot{\mathbf{R}}^T\mathbf{x}'\right) = \mathbf{R}\dot{\mathbf{e}} \qquad [2.79]$$

having used, in order: the identity $\mathbf{v}' \equiv \dot{\mathbf{x}}'$, equations [2.58] and [2.59], time-differentiation of equation [2.4] and, finally, equation [2.24]. In summary, *the stretching velocity gradient is equal to the strain-rate vector pushed-forward*, i.e.:

$$\mathbf{d} := \mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}' = \mathbf{R}\dot{\mathbf{e}} \qquad [2.80]$$

In other words, $\mathbf{d}$ and $\dot{\mathbf{e}}$ describe the same physical quantity, namely the increment of strain accumulated in an infinitesimal time interval $dt = 1$; however, they are seen by two different observers, attached to the bases $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively. Therefore, $\mathbf{d}$ has in the current $\mathcal{B}$-basis the same components that $\dot{\mathbf{e}}$ has in the reference $\bar{\mathcal{B}}$-basis.

A geometrical interpretation of this property is given in Figure 2.8, which concerns an example of local pure extension and rotation around $\bar{\mathbf{a}}_3$. The figure shows the segment $\overline{PQ} = 1$ in the reference configuration (at $t = 0$), in the current configuration (at $t > 0$), and in a configuration adjacent to the latter (at $t + dt$, with $dt = 1$). In Figure 2.8(a) the decomposition of the velocity gradient $\mathbf{g} = \dot{\mathbf{x}}'$, in its stretching $\mathbf{d}$ and rigid part $\mathbf{w} = \boldsymbol{\omega} \times \mathbf{x}'$, is shown. In Figure 2.8(b) the strain measures $\mathbf{e}_r, \mathbf{e}_c$ and the strain-rates $\dot{\mathbf{e}}_r, \dot{\mathbf{e}}_c$ are illustrated. It is apparent that according to equation [2.79], $\mathbf{d} = \mathbf{R}\dot{\mathbf{e}}_r$. This relation can be interpreted as follows: (a) the current strain $\mathbf{e}_c$ is first pulled-back as $\mathbf{e}_r = \mathbf{R}^T\mathbf{e}_c$ in the reference configuration; (b) here it is time-differentiated, to become $\dot{\mathbf{e}}_r$; (c) finally, this is further pushed-forward as $\mathbf{R}\dot{\mathbf{e}}_r$, which coincides with $\mathbf{d}$. In contrast, $\dot{\mathbf{e}}_c \neq \mathbf{d}$, since, by the Poisson formula [2.64], $\dot{\mathbf{e}}_c = \mathbf{R}\dot{\mathbf{e}}_c + \boldsymbol{\omega} \times \mathbf{e}_c$, as highlighted in the figure. For these reasons, the reference measure of the strain appears better suited than the current one in the framework of our formulation.

Similar to what was done for the strain, we now compare the curvature-rate $\dot{\mathbf{k}}$ and the spin gradient $\boldsymbol{\omega}'$. We start from the definitions of $\mathbf{k}$ and $\boldsymbol{\omega}$, given in equations [2.44] and [2.59], respectively, and we differentiate the first one with respect to s and the second one with respect to t by getting:

$$\begin{aligned} &\dot{\mathbf{R}}^T\mathbf{a}_j' + \mathbf{R}^T\dot{\mathbf{a}}_j' = \dot{\mathbf{k}} \times \bar{\mathbf{a}}_j \\ &\dot{\mathbf{a}}_j' = \boldsymbol{\omega}' \times \mathbf{a}_j + \boldsymbol{\omega} \times \mathbf{a}_j' \end{aligned} \qquad [2.81]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

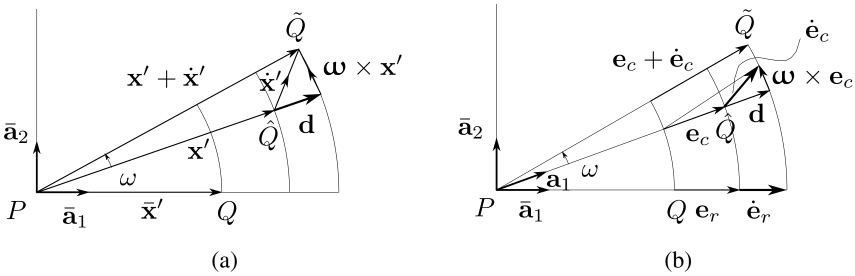


Figure 2.8: Geometrical interpretation of: (a) the stretching velocity gradient $\mathbf{d}$, (b) the strain rate vectors $\dot{\mathbf{e}}_r$ and $\dot{\mathbf{e}}_c$. The example concerns pure extension of the beam, and a rotation around the fixed axis $\bar{\mathbf{a}}_3$; moreover, $ds = dt = 1$.

The first of them can be rearranged as follows:

$$\begin{aligned} \dot{\mathbf{a}}'_j &= -\mathbf{R}\dot{\mathbf{R}}^T \mathbf{a}'_j + \mathbf{R}\left(\dot{\mathbf{k}} \times \bar{\mathbf{a}}_j\right) \\ &= \mathbf{W}\mathbf{a}'_j + \mathbf{R}\dot{\mathbf{k}} \times \mathbf{R}\bar{\mathbf{a}}_j \\ &= \boldsymbol{\omega} \times \mathbf{a}'_j + \mathbf{R}\dot{\mathbf{k}} \times \mathbf{a}_j \end{aligned} \qquad [2.82]$$

in which $\mathbf{R}\dot{\mathbf{R}}^T = \mathbf{W}^T = -\mathbf{W}$ and a known identity has been used[22]. If we compare the last equation with equation [2.81b], we finally get:

$$\mathbf{s} := \boldsymbol{\omega}' = \mathbf{R}\dot{\mathbf{k}} \qquad [2.83]$$

which is the companion of equation [2.79]. Once again, a stretching gradient coincides with a pushed-forward reference strain-rate; therefore, previous remarks still hold.

Strain-rate-velocity scalar relationships

Our task is now writing *strain-rates* in scalar form, in the reference basis. From equations [2.80] and [2.83], we have:

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{R}^T \left(\mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}'\right) \\ \dot{\mathbf{k}} &= \mathbf{R}^T \boldsymbol{\omega}' \end{aligned} \qquad [2.84]$$

We rewrite the first of them by noting that the cross-product can be expressed as the product of a skew-symmetric tensor by a vector; namely $\boldsymbol{\omega} \times \mathbf{x}' = \mathbf{W}\mathbf{x}'$ (as we are

22. The identity $\mathbf{R}\mathbf{u} \times \mathbf{R}\mathbf{v} = \mathbf{R}\left(\mathbf{u} \times \mathbf{v}\right)$ expresses "the rotational invariance of cross product". It means that, if $\mathbf{u}$ and $\mathbf{v}$ are both rotated by $\mathbf{R}$, then their cross-product is also rotated by $\mathbf{R}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

used to doing), but also $\mathbf{x}' \times \boldsymbol{\omega} = \boldsymbol{\Lambda}\boldsymbol{\omega}$. Here, $\boldsymbol{\Lambda}$ is a tensor[23] whose axial vector is $\mathbf{x}'$ (see equation [2.28]) whose representation in the current basis is:

$$\boldsymbol{\Lambda} := \begin{bmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -(1+\varepsilon) \\ -\gamma_2 & 1+\varepsilon & 0 \end{bmatrix} \quad [2.85]$$

since, from equation [2.24], $\mathbf{x}' = \mathbf{R}(\mathbf{e} + \bar{\mathbf{a}}_1)$. Therefore:

$$[\mathbf{x}' \times \boldsymbol{\omega}]_{\bar{\mathcal{B}}} = \boldsymbol{R}[\mathbf{x}' \times \boldsymbol{\omega}]_{\mathcal{B}} = \boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{\omega} = \boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{R}^T\bar{\boldsymbol{\omega}} \quad [2.86]$$

where $\boldsymbol{\omega} = \boldsymbol{R}^T\bar{\boldsymbol{\omega}}$ has been used. By letting $\dot{\mathbf{e}} = \dot{\varepsilon}\bar{\mathbf{a}}_1 + \dot{\gamma}_2\bar{\mathbf{a}}_2 + \dot{\gamma}_3\bar{\mathbf{a}}_3$, $\dot{\mathbf{k}} = \dot{\kappa}_1\bar{\mathbf{a}}_1 + \dot{\kappa}_2\bar{\mathbf{a}}_2 + \dot{\kappa}_3\bar{\mathbf{a}}_3$ and collecting the strain-rate components in the column-matrices $\dot{\boldsymbol{e}}$, $\dot{\boldsymbol{k}}$, we finally have the scalar representation of equations [2.84] in the reference basis:

$$\begin{aligned} \dot{\boldsymbol{e}} &= \boldsymbol{R}^T\boldsymbol{v}' + \boldsymbol{\Lambda}\boldsymbol{R}^T\bar{\boldsymbol{\omega}} \\ \dot{\boldsymbol{k}} &= \boldsymbol{R}^T\bar{\boldsymbol{\omega}}' \end{aligned} \quad [2.87]$$

where $\boldsymbol{v} := (\dot{u}_1, \dot{u}_2, \dot{u}_3)^T$, $\bar{\boldsymbol{\omega}} := (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)^T$. These are called the *strain-rate-velocity relationships*.

Equations [2.87] link the strain-rates to the angular velocity $\bar{\boldsymbol{\omega}}$; however, we will fulfill the need to express them in terms of the time-derivatives $\dot{\boldsymbol{\theta}}$. Within this scope, by using $\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{B}}_\omega\dot{\boldsymbol{\theta}}$ (equation [2.71a]), we also find [24]:

$$\begin{aligned} \dot{\boldsymbol{e}} &= \boldsymbol{R}^T\boldsymbol{v}' + \boldsymbol{\Lambda}\boldsymbol{R}^T\bar{\boldsymbol{B}}_\omega\dot{\boldsymbol{\theta}} \\ \dot{\boldsymbol{k}} &= \boldsymbol{R}^T\left(\bar{\boldsymbol{B}}_\omega'\dot{\boldsymbol{\theta}} + \bar{\boldsymbol{B}}_\omega\dot{\boldsymbol{\theta}}'\right) \end{aligned} \quad [2.88]$$

2.2 Dynamics

After having analyzed the kinematic aspects of the beam model, we have to deal with dynamic features. As we saw in section 1.2.2, dynamics is ruled by balance

23. The symbol $\boldsymbol{\Lambda}$ adopted should evoke the stretch $\lambda := dx/ds$, used in Continuum Mechanics.

24. When equations [2.88] are put in matrix form, and $\boldsymbol{v} = \dot{\boldsymbol{u}}$ is used, we have

$$\begin{pmatrix} \dot{\boldsymbol{e}} \\ \dot{\boldsymbol{k}} \end{pmatrix} = \left(\begin{bmatrix} \mathbf{0} & \boldsymbol{\Lambda}\boldsymbol{R}^T\bar{\boldsymbol{B}}_\omega \\ \mathbf{0} & \boldsymbol{R}^T\bar{\boldsymbol{B}}_\omega' \end{bmatrix} + \begin{bmatrix} \boldsymbol{R}^T & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}^T\bar{\boldsymbol{B}}_\omega \end{bmatrix} \partial_s \right) \begin{pmatrix} \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{\theta}} \end{pmatrix}$$

They look like equations [1.3], that we wrote for the metamodel. Indeed, if we collect the strain-rate components and velocities in column-matrices $\boldsymbol{\varepsilon} := (\varepsilon, \gamma_2, \gamma_3, \kappa_1, \kappa_2, \kappa_3)^T$, $\boldsymbol{w} := (u_1, u_2, u_3, \theta_1, \theta_2, \theta_3)^T$, the equation above becomes $\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}(\boldsymbol{w}, \boldsymbol{w}')\dot{\boldsymbol{w}}$, where $\boldsymbol{D}(\boldsymbol{w}, \boldsymbol{w}')$ is the 6×6 *kinematic operator* of the straight beam.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

equations, whose derivation calls for introducing stresses and linking them to external forces by invoking a principle. The task can be alternatively accomplished by invoking two different *balance principles*: (a) the VPP or (b) the linear and angular momentum principles. According to the former, a balance of powers is performed; according to the latter, a balance of forces is executed. Both of them are integral principles, but their localization leads to differential equations. We will illustrate both the procedures.

The object of our study is a beam loaded by external forces of linear density $\mathbf{p}(s,t)$ (having physical dimensions $[\mathrm{MT}^{-2}]$) and external couples of linear density $\mathbf{c}(s,t)$ ($[\mathrm{MLT}^{-2}]$) represented by a vector orthogonal to the plane of the couple (see Figure 2.9); moreover, forces $\mathbf{P}_H(t)$ ($[\mathrm{MLT}^{-2}]$) and couples $\mathbf{C}_H(t)$ ($[\mathrm{ML}^2\mathrm{T}^{-2}]$) are applied at the ends $H = A, B$.

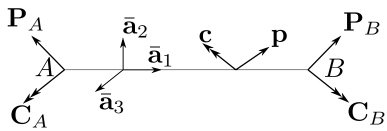


Figure 2.9: Beam loaded by forces and couples, both distributed in the body and applied at the ends.

2.2.1 The balance of virtual powers

The external and internal virtual powers

We consider the beam frozen in the *unknown* current configuration, occupied at time t and superimpose a *virtual motion* consisting of a velocity field $\mathbf{v}$ and a spin field $\boldsymbol{\omega}$. These fields are also called a *test motion*, since they are arbitrary and completely unrelated to the true motion of the beam. We say that the virtual motion is *kinematically admissible* when it is sufficiently regular and compatible with the external constraints; if the latter depend on time, they must be frozen at t[25]. Then, a velocity gradient $\mathbf{g} = \mathbf{d} + \mathbf{w}$ and a spin gradient $\mathbf{s}$ can be associated with it.

25. For example, if the displacement is prescribed, i.e. $\mathbf{u}_H = \breve{\mathbf{u}}_H(t)$, then $\dot{\mathbf{u}}_H = \mathbf{0}$; similarly, if the rotation is assigned, $\mathbf{R}_H = \breve{\mathbf{R}}_H(t)$, then $\boldsymbol{\omega}_H = \mathbf{0}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

We define the *external virtual power* according to the usual definition given for a system of forces[26]:

$$\mathcal{P}_{ext} := \int_{\mathcal{S}} (\mathbf{p} \cdot \mathbf{v} + \mathbf{c} \cdot \boldsymbol{\omega})\, ds + \sum_{H=A}^{B} (\mathbf{P}_H \cdot \mathbf{v}_H + \mathbf{C}_H \cdot \boldsymbol{\omega}_H) \qquad [2.89]$$

Then, in a similar way, we define the *internal virtual power* as the power spent by *contact internal forces* (or stresses) $\mathbf{t}, \mathbf{m}$, *on the stretching velocity gradient* $\mathbf{d} = \mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}'$ and the *spin gradient* $\mathbf{s} = \boldsymbol{\omega}'$, namely:

$$\mathcal{P}_{int} := \int_{\mathcal{S}} (\mathbf{t} \cdot \mathbf{d} + \mathbf{m} \cdot \mathbf{s})\, ds \qquad [2.90]$$

The vector $\mathbf{t}$ is called the *force-stress*, and the vector $\mathbf{m}$ the *couple-stress*[27]. The definition [2.90] meets the *principle of material frame-indifference* (or *of objectivity*), which requires that $\mathcal{P}_{int} = 0$ for any rigid motion. As a matter of fact, in a rigid motion, $\mathbf{d} = \mathbf{0}$ and $\mathbf{s} = \mathbf{0}$, as we commented in section 2.1.8. Therefore, the stresses are dynamic entities which spend zero virtual power in rigid motions[28].

A more formal derivation of the expression of the internal virtual power is the following. We start from a general definition, in which the internal virtual power is a linear combination of dynamic quantities and kinematic descriptors taken as the velocity fields $\mathbf{v}, \boldsymbol{\omega}$ and their first derivatives $\mathbf{v}', \boldsymbol{\omega}'$, namely:

$$\mathcal{P}_{int} := \int_{\mathcal{S}} \left(\mathbf{t}_0 \cdot \mathbf{v} + \mathbf{t}_1 \cdot \mathbf{v}' + \mathbf{m}_0 \cdot \boldsymbol{\omega} + \mathbf{m}_1 \cdot \boldsymbol{\omega}'\right) ds \qquad [2.91]$$

where $\mathbf{t}_0$ is the (internal) *self-force*, $\mathbf{t}_1$ is the *force*, $\mathbf{m}_0$ is the *self-couple* and $\mathbf{m}_1$ is the *couple*. The four stresses, however, are not all independent, since $\mathcal{P}_{int}$ must vanish when the test motion is rigid. In this occurrence, $\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times (\mathbf{x} - \mathbf{x}_0)\,,\ \boldsymbol{\omega} = \boldsymbol{\omega}_0$, with $\mathbf{v}_0, \boldsymbol{\omega}_0$ constant on $\mathcal{S}$, from which $\mathbf{v}' = \boldsymbol{\omega}_0 \times \mathbf{x}',\ \boldsymbol{\omega}' = \mathbf{0}$. Therefore, the principle requires that:

$$\int_{\mathcal{S}} \left[\mathbf{t}_0 \cdot \mathbf{v}_0 + \left((\mathbf{x} - \mathbf{x}_0) \times \mathbf{t}_0 + \mathbf{x}' \times \mathbf{t}_1 + \mathbf{m}_0\right) \cdot \boldsymbol{\omega}_0\right] ds = 0 \qquad \forall\, (\mathbf{v}_0, \boldsymbol{\omega}_0) \quad [2.92]$$

from which follows:

$$\mathbf{t}_0 = \mathbf{0}, \qquad \mathbf{m}_0 = -\mathbf{x}' \times \mathbf{t}_1 \qquad [2.93]$$

---

26. The adjective “virtual”, here and in the following, stresses the fact that the dynamical and kinetic quantities are unrelated.

27. Note that (as for the Cauchy continuum) just the first gradient of the velocity appears in the definition of the internal virtual power. For this reason, the theory is of *first gradient* (or degree-1).

28. This definition is equivalent to that given in the force balance approach (section 1.2.2), according to which the stress is the more general system of self-equilibrated internal forces acting on an infinitesimal element of the body.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

In conclusion, with these restrictions, the internal virtual power [2.91] assumes the form:

$$\mathcal{P}_{int} := \int_{\mathcal{S}} \left[\mathbf{t}_1 \cdot \left(\mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}' \right) + \mathbf{m}_1 \cdot \boldsymbol{\omega}' \right] ds \qquad [2.94]$$

which is just equation [2.90], with the index 1 omitted on the stresses.

REMARK 2.12. Equation [2.90], which defines the internal virtual power, implicitly also defines the stresses. These, however, are *not* introduced on the ground of a dynamic characterization of the internal contact interactions, as will be done in the equilibrium approach, but rather are defined as (not observable) *dual* quantities of kinetic (observable) quantities.

The stress components

The stresses $\mathbf{t}, \mathbf{m}$ are conveniently expressed in the current configuration (Figure 2.10(b) as:

$$\begin{aligned} \mathbf{t} &= N\mathbf{a}_1 + T_2\mathbf{a}_2 + T_3\mathbf{a}_3 \\ \mathbf{m} &= M_1\mathbf{a}_1 + M_2\mathbf{a}_2 + M_3\mathbf{a}_3 \end{aligned} \qquad [2.95]$$

Here, N is the *normal force*[29]; T_2, T_3 are the *shear forces*; M_1 is the *twisting moment* and M_2, M_3 are *the bending moments*. Therefore, the state of stress depends on six scalar fields.

The internal virtual power in terms of strain-rates

The internal virtual power has been introduced in terms of stretching velocity gradients. However, it is possible to give to it an alternative expression in terms of strain-rates that will be useful later.

We remember (equations [2.80] and [2.83]) that the stretching velocity gradients are related to the strain-rates via $\mathbf{d} = \mathbf{R}\dot{\mathbf{e}}$ and $\mathbf{s} = \mathbf{R}\dot{\mathbf{k}}$; therefore, equation [2.90] can also be written as:

$$\mathcal{P}_{int} := \int_{\mathcal{S}} \left(\mathbf{t} \cdot \mathbf{R}\dot{\mathbf{e}} + \mathbf{m} \cdot \mathbf{R}\dot{\mathbf{k}} \right) ds \qquad [2.96]$$

29. Here, according to the tradition, allusion is made to the section, although it disappeared in the polar model.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Both the strain-rates, $\dot{\mathbf{e}}, \dot{\mathbf{k}}$, refer to the $\bar{\mathcal{B}}$-basis, but pre-multiplication by $\mathbf{R}$ pushes them to the $\mathcal{B}$-basis, according to:

$$\begin{aligned} \mathbf{R}\dot{\mathbf{e}} &= \dot{\varepsilon}\mathbf{a}_1 + \dot{\gamma}_2\mathbf{a}_2 + \dot{\gamma}_3\mathbf{a}_3 \\ \mathbf{R}\dot{\mathbf{k}} &= \dot{\kappa}_1\mathbf{a}_1 + \dot{\kappa}_2\mathbf{a}_2 + \dot{\kappa}_3\mathbf{a}_3 \end{aligned} \qquad [2.97]$$

so that, by using equations [2.95], which express stresses in $\mathcal{B}$, we finally have:

$$\mathcal{P}_{int} = \int_{\mathcal{S}} (N\dot{\varepsilon} + T_2\dot{\gamma}_2 + T_3\dot{\gamma}_3 + M_1\dot{\kappa}_1 + M_2\dot{\kappa}_2 + M_3\dot{\kappa}_3)\, ds \qquad [2.98]$$

Therefore, the *internal virtual power is sum of the products between the stress components, evaluated in the current basis, and the dual strain-rate components, evaluated in the reference basis*[30].

The Virtual Power Principle

The VPP establishes that *in any kinematically admissible virtual motion*, *the external virtual power spent by the forces* $\mathbf{p}, \mathbf{c}, \mathbf{P}_H, \mathbf{C}_H$ *on the velocity and spin fields* $\mathbf{v}, \boldsymbol{\omega}$, *equates the internal virtual power spent by the stresses* $\mathbf{t}, \mathbf{m}$ *on the stretching velocity gradient fields* $\mathbf{d}, \mathbf{s}$, i.e.:

$$\mathcal{P}_{ext} = \mathcal{P}_{int} \qquad \forall\, (\mathbf{v}, \boldsymbol{\omega}) \qquad [2.99]$$

or by remembering that $\mathbf{d} = \mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}'$ and $\mathbf{s} = \boldsymbol{\omega}'$:

$$\begin{aligned} &\int_{\mathcal{S}} (\mathbf{p} \cdot \mathbf{v} + \mathbf{c} \cdot \boldsymbol{\omega}) ds + \sum_{H=A}^{B} (\mathbf{P}_H \cdot \mathbf{v}_H + \mathbf{C}_H \cdot \boldsymbol{\omega}) \\ &\quad = \int_{\mathcal{S}} [\mathbf{t} \cdot (\mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}') + \mathbf{m} \cdot \boldsymbol{\omega}']\, ds \qquad \forall\, (\mathbf{v}, \boldsymbol{\omega}) \end{aligned} \qquad [2.100]$$

30. With the symbols of the metamodel (equation [1.6]), it is $\mathcal{P}_{int} = \int_{\mathcal{S}} \boldsymbol{\sigma}^T \dot{\boldsymbol{\varepsilon}} ds$, with $\boldsymbol{\sigma} := (N, T_2, T_3, M_1, M_2, M_3)^T$ and $\dot{\boldsymbol{\varepsilon}} := (\dot{\varepsilon}, \dot{\gamma}_2, \dot{\gamma}_3, \dot{\kappa}_1, \dot{\kappa}_2, \dot{\kappa}_3)^T$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The balance equations and boundary conditions

The VPP provides the balance equations via the following procedure. By integrating by parts and permuting the factor of the mixed product, we have:

$$\int_{\mathcal{S}} (\mathbf{p}\cdot\mathbf{v}+\mathbf{c}\cdot\boldsymbol{\omega})\,ds+\sum_{H=A}^{B}(\mathbf{P}_H\cdot\mathbf{v}_H+\mathbf{C}_H\cdot\boldsymbol{\omega}_H)$$
$$=-\int_{\mathcal{S}}(\mathbf{t}'\cdot\mathbf{v}+\mathbf{m}'\cdot\boldsymbol{\omega}+\mathbf{x}'\times\mathbf{t}\cdot\boldsymbol{\omega})\,ds+[\mathbf{t}\cdot\mathbf{v}+\mathbf{m}\cdot\boldsymbol{\omega}]_A^B \qquad \forall(\mathbf{v},\boldsymbol{\omega}) \tag{2.101}$$

or, rearranging:

$$\int_{\mathcal{S}}[(\mathbf{t}'+\mathbf{p})\cdot\mathbf{v}+(\mathbf{m}'+\mathbf{x}'\times\mathbf{t}+\mathbf{c})\cdot\boldsymbol{\omega}]\,ds$$
$$+\sum_{H=A}^{B}[(\mathbf{P}_H\pm\mathbf{t}_H)\cdot\mathbf{v}_H+(\mathbf{C}_H\pm\mathbf{m}_H)\cdot\boldsymbol{\omega}_H]=0 \qquad \forall(\mathbf{v},\boldsymbol{\omega}) \tag{2.102}$$

where the upper sign holds at A and the lower at B. Due to the arbitrariness of the test motion, the following field equations are derived:

$$\begin{aligned}&\mathbf{t}'+\mathbf{p}=\mathbf{0}\\&\mathbf{m}'+\mathbf{x}'\times\mathbf{t}+\mathbf{c}=\mathbf{0}\end{aligned} \tag{2.103}$$

together with the boundary conditions:

$$\begin{aligned}&(\mathbf{P}_H\pm\mathbf{t}_H)\cdot\mathbf{v}_H=0\\&(\mathbf{C}_H\pm\mathbf{m}_H)\cdot\boldsymbol{\omega}_H=0, \qquad H=A,B\end{aligned} \tag{2.104}$$

Equations [2.103] are the (*local form of the*) *balance equations*. Equations [2.104] generate the relevant boundary conditions. Once the geometric (or essential) boundary conditions [2.17] are prescribed, they supply the *natural conditions*. For example, (a) if H is clamped, then $\mathbf{v}_H=\mathbf{0},\ \boldsymbol{\omega}_H=\mathbf{0}$, and therefore no mechanical condition must be enforced there; (b) if H is free, then $\mathbf{v}_H\neq\mathbf{0},\ \boldsymbol{\omega}_H\neq\mathbf{0}$ are arbitrary, and therefore $\mathbf{P}_H\pm\mathbf{t}_H=\mathbf{0},\ \mathbf{C}_H\pm\mathbf{m}_H=\mathbf{0}$ must hold there. Therefore, geometric and mechanical conditions are alternative.

The same property holds for partially restrained ends. For example, if the end H is constrained by a spherical hinge, then $\mathbf{v}_H=\mathbf{0}$ but $\boldsymbol{\omega}_H$ is arbitrary so that the mechanical conditions only concern the couple-stress, i.e. $\mathbf{C}_H\pm\mathbf{m}_H=\mathbf{0}$. As a second example, if the constraint is a cylindrical hinge of axis $\bar{\mathbf{n}}$, then $\mathbf{v}_H=\mathbf{0}$ and $\boldsymbol{\omega}_H=\dot{\Theta}\bar{\mathbf{n}}$, where the angular velocity $\dot{\Theta}$ is arbitrary. Then, the mechanical condition is scalar, i.e. $(\mathbf{C}_H\pm\mathbf{m}_H)\cdot\bar{\mathbf{n}}=0$, and expresses the equality of the couple-stress and the external couple, when they are both projected along the fixed rotation axis $\bar{\mathbf{n}}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2.2.2 The inertial contributions

We now account for inertial effects on the dynamics of the beam. We will first incorporate the inertia forces in the VPP, and then we will illustrate an alternative approach, based on the Extended Hamilton Principle (EHP).

The inertia forces

If dynamical effects are not negligible, by invoking the d'Alembert principle, the inertia forces $\mathbf{p}^{in} := -m\dot{\mathbf{v}}$ and the inertia couples $\mathbf{c}^{in} := -\mathbf{I}_G\dot{\boldsymbol{\omega}}$ must be included in the external power, with m the mass per unit length of the beam and $\mathbf{I}_G$ the inertia tensor of the cross-section (we will derive later from the 3D-model, see equation [2.113]). Therefore, the VPP [2.99] modifies into:

$$\mathcal{P}_{ext}^{act} + \mathcal{P}_{ext}^{in} = \mathcal{P}_{int} \qquad [2.105]$$

where[31]:

$$\begin{aligned} \mathcal{P}_{ext}^{act} &:= \int_{\mathcal{S}} (\mathbf{p}\cdot\hat{\mathbf{v}} + \mathbf{c}\cdot\hat{\boldsymbol{\omega}})\, ds + \sum_{H=A}^{B} (\mathbf{P}_H\cdot\hat{\mathbf{v}}_H + \mathbf{C}_H\cdot\hat{\boldsymbol{\omega}}_H) \\ \mathcal{P}_{ext}^{in} &:= \int_{\mathcal{S}} (-\mathbf{m}\dot{\mathbf{v}}\cdot\hat{\mathbf{v}} - \mathbf{I}_G\dot{\boldsymbol{\omega}}\cdot\hat{\boldsymbol{\omega}})\, ds \end{aligned} \qquad [2.106]$$

Here, a hat (not necessary before) has been used to denote virtual velocities $\hat{\mathbf{v}}, \hat{\boldsymbol{\omega}}$, in order to not confuse them with the *unrelated* true velocities $\mathbf{v}, \boldsymbol{\omega}$. Accordingly, the balance equations become:

$$\begin{aligned} &\mathbf{t}' + \mathbf{p} = m\ddot{\mathbf{u}} \\ &\mathbf{m}' + \mathbf{x}' \times \mathbf{t} + \mathbf{c} = \mathbf{I}_G\dot{\boldsymbol{\omega}} \end{aligned} \qquad [2.107]$$

The Extended Hamilton Principle

An alternative derivation of the principle [2.105], and the consequent balance equations [2.107], descends from the EHP, (see equation [1.108]). It works as a generalization of the VPP and avoids the need to resort to the notion of inertia force.

31. A boundary term in $\mathcal{P}_{ext}^{in}$ can also exist, if lumped masses, not considered here, are attached at the ends of the beam.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The EHP, in its usual form, states that:

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W)\, dt = 0, \quad \forall (\delta \mathbf{u}, \delta \mathbf{R})\, | \delta \mathbf{u}(s, t_i) = \mathbf{0},\ \delta \mathbf{R}(s, t_i) = \mathbf{0},\ i = 1, 2$$

[2.108]

where δT is the variation of kinetic energy, δU the work spent by the internal forces and δW the work spent by the external forces during the motion in the interval δt. It can be recast in terms of powers as follows:

$$\int_{t_1}^{t_2} \left(\frac{\delta T}{\delta t} - \frac{\delta U}{\delta t} + \frac{\delta W}{\delta t} \right) dt = 0, \quad \forall (\hat{\mathbf{v}}, \hat{\boldsymbol{\omega}})\, | \hat{\mathbf{v}}(s, t_i) = \mathbf{0},\ \hat{\boldsymbol{\omega}}(s, t_i) = \mathbf{0},\ i = 1, 2$$

[2.109]

But, according to equation [2.106a], $\delta W / \delta t \equiv \mathcal{P}_{ext}^{act}$; moreover, according to equation [2.90], or to the alternative expression [2.98], it is $\delta U / \delta t \equiv \mathcal{P}_{int}$. Therefore, in order to reobtain equation [2.105], it only needs to prove that $\delta T / \delta t \equiv \mathcal{P}_{ext}^{in}$. To this end, we start from the definition of kinetic energy:

$$T := \int_{\mathcal{S}} \left(\frac{1}{2} m \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \mathbf{I}_G \boldsymbol{\omega} \cdot \boldsymbol{\omega} \right) ds \qquad [2.110]$$

from which:

$$\delta T = \int_{\mathcal{S}} \left(m \mathbf{v} \cdot \delta \hat{\mathbf{v}} + \mathbf{I}_G \boldsymbol{\omega} \cdot \delta \hat{\boldsymbol{\omega}} \right) ds \qquad [2.111]$$

and finally:

$$\frac{\delta T}{\delta t} = \int_{\mathcal{S}} \left(m \mathbf{v} \cdot \frac{\delta \hat{\mathbf{v}}}{\delta t} + \mathbf{I}_G \boldsymbol{\omega} \cdot \frac{\delta \hat{\boldsymbol{\omega}}}{\delta t} \right) ds = \int_{\mathcal{S}} \left(-m \dot{\mathbf{v}} \cdot \hat{\mathbf{v}} - \mathbf{I}_G \dot{\boldsymbol{\omega}} \cdot \hat{\boldsymbol{\omega}} \right) ds = \mathcal{P}_{ext}^{in}$$

[2.112]

where an integration by parts has been performed and boundary terms vanished, as requested in equation [2.109].

In conclusion, $\delta T / \delta t$ accounts for the contribution of the d'Alembert inertia forces to the balance equations.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The inertia tensor

The inertia tensor $\mathbf{I}_G$ (whose dimensions are $[\mathrm{ML}]$), appearing in the inertia forces and in the kinetic energy, can be derived by the 3D-model of beam with rigid cross-section. In the *current basis*, it assumes the following representation:

$$\boldsymbol{I}_G := [\mathbf{I}_G]_{\mathcal{B}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & -I_{23} \\ 0 & -I_{23} & I_3 \end{pmatrix} \qquad [2.113]$$

where:

$$\begin{aligned} I_1 &:= \int_{\mathcal{A}} \varrho \left(r_2^2 + r_3^2\right) dA, \qquad & I_2 &:= \int_{\mathcal{A}} \varrho r_3^2 dA, \\ I_3 &:= \int_{\mathcal{A}} \varrho r_2^2 dA, & I_{23} &:= \int_{\mathcal{A}} \varrho r_2 r_3 dA \end{aligned} \qquad [2.114]$$

where ϱ is the mass density $\left[\mathrm{ML}^{-3}\right]$, $\mathcal{A}$ is the cross-section and dA is the element of area. If the triad $\mathcal{B}$ is *principal of inertia*, then the mixed moment of inertia vanishes, $I_{23} = 0$, so that the tensor becomes diagonal. We will later refer to this simplest case.

To prove equation [2.113], we write the kinetic energy for unit length of the 3D-beam:

$$\begin{aligned} \frac{dT}{ds} :=& \frac{1}{2} \int_{\mathcal{A}} \varrho \mathbf{v}_Q \cdot \mathbf{v}_Q dA \\ =& \frac{1}{2} \int_{\mathcal{A}} \varrho \|\mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}\|^2 dA \\ =& \frac{1}{2} \int_{\mathcal{A}} \varrho \left(\|\mathbf{v}_G\|^2 + \|\boldsymbol{\omega} \times \mathbf{r}\|^2\right) dA \end{aligned} \qquad [2.115]$$

where we used equation [2.69] and accounted for the $\int_{\mathcal{A}} \mathbf{r} dA = \mathbf{0}$. Since:

$$\begin{aligned} \|\boldsymbol{\omega} \times \mathbf{r}\|^2 &= \left(\|\boldsymbol{\omega}\|\|\mathbf{r}\| \sin \widehat{(\boldsymbol{\omega}, \mathbf{r})}\right)^2 \\ &= \|\boldsymbol{\omega}\|^2 \|\mathbf{r}\|^2 \left[1 - \left(\frac{\boldsymbol{\omega} \cdot \mathbf{r}}{\|\boldsymbol{\omega}\|\|\mathbf{r}\|}\right)^2\right] \\ &= (\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r} \cdot \mathbf{r}) - (\boldsymbol{\omega} \cdot \mathbf{r})^2 \end{aligned} \qquad [2.116]$$

by defining:

$$m := \int_{\mathcal{A}} \rho dA, \qquad \mathbf{I}_G \boldsymbol{\omega} := \int_{\mathcal{A}} \varrho \left[(\mathbf{r} \cdot \mathbf{r}) \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r}) \mathbf{r}\right] dA \qquad [2.117]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

it follows that $\frac{dT}{ds} = \frac{1}{2}m\mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2}\mathbf{I}_G\boldsymbol{\omega} \cdot \boldsymbol{\omega}$, i.e. equation [2.110] is derived, with $\mathbf{v} \equiv \mathbf{v}_G$.

By expanding the products in the definition of the inertia tensor we have:

$$\begin{aligned}\mathbf{I}_G\boldsymbol{\omega} &= \int_{\mathcal{A}} \left[\varrho\left(r_2^2 + r_3^2\right)(\omega_1\mathbf{a}_1 + \omega_2\mathbf{a}_2 + \omega_3\mathbf{a}_3)\right.\\ &\quad \left. - (r_2\omega_2 + r_3\omega_3)(r_2\mathbf{a}_2 + r_3\mathbf{a}_3)\right] dA\\ &= \int_{\mathcal{A}} \varrho\left[\left(r_2^2 + r_3^2\right)\omega_1\mathbf{a}_1 + \left(r_3^2\omega_2 - r_2r_3\omega_3\right)\mathbf{a}_2\right.\\ &\quad \left. + \left(r_2^2\omega_3 - r_2r_3\omega_2\right)\mathbf{a}_3\right] dA\end{aligned} \qquad [2.118]$$

from which equation [2.113] follows.

2.2.3 The balance of momentum

An alternative procedure to derive the local balance equations is offered by the principles of the (linear) *momentum* and of *the angular moment*, which lead to equations known as “cardinal equations of dynamics”, or, in the static case, “equilibrium equations”, or “balance of force equations”. These principles are postulates of continuum mechanics, borrowed from mechanics of a collection of particles and rigid bodies. They state that “the time-derivative of the total (angular) momentum of *any* parts of a body is equal to the vector sum (of the moments) of the external forces acting on it”. Since a part of the body is subjected to external and internal forces (or stresses), we have to first define stresses on a physical ground, which is something different from the definition we gave before via the concept of internal power. We will see, however, that the two approaches give the same results.

The linear and angular momentum

According to well-known results of rigid-body dynamics, we introduce two vector quantities:

$$\mathbf{j} := m\mathbf{v} \qquad [2.119]$$

called the *momentum* per unit length of the beam[32], and:

$$\mathbf{h} := \mathbf{x} \times m\mathbf{v} + \mathbf{I}_G\boldsymbol{\omega} \qquad [2.120]$$

32. The linear momentum is often denoted by $\mathbf{p}$ or $\mathbf{f}$; these symbols, however, assume different meanings in this book.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

called the *angular momentum* per unit length of the beam (with respect to a fixed pole O). Here, m is the mass per unit length of the beam, and $\mathbf{I}_G$ is the *mass inertia tensor* of the cross-section with respect to the center of mass G defined in equation [2.113]. The two terms on the right side of equation [2.120] are known as *translational* and *rotational* contribution to the angular moment, respectively.

We can derive the previous expressions by resorting to a 3D-model of a beam with rigid cross-sections. Let us consider the cross-section at abscissa s, and let G be its centroid, that we will assume coincident with its center of mass (as it happens, for example, when the mass density is constant). The momentum per unit length of the beam is defined as:

$$\mathbf{j} := \int_{\mathcal{A}} \varrho \mathbf{v}_Q dA = \int_{\mathcal{A}} \varrho \mathbf{v}_G dA + \int_{\mathcal{A}} \varrho \boldsymbol{\omega} \times \mathbf{r} dA \qquad [2.121]$$

where the "Fundamental Formula of Rigid Kinematics", has been used (equation [2.69]). Since G is center of mass of the section, the second integral (being linear in $\mathbf{r}$) vanishes, and equation [2.119] is reobtained, in which $m := \int_{\mathcal{A}} \varrho dA$, and $\mathbf{v} := \mathbf{v}_G$ has been posed.

The angular momentum per unit length of the beam with respect to O is defined as:

$$\begin{aligned} \mathbf{h} :=& \int_{\mathcal{A}} \varrho \mathbf{x}_Q \times \mathbf{v}_Q dA \\ =& \mathbf{x}_G \times \mathbf{v}_G \int_{\mathcal{A}} \varrho dA + \mathbf{x}_G \times \int_{\mathcal{A}} \varrho \left(\boldsymbol{\omega} \times \mathbf{r}\right) dA \\ &+ \int_{\mathcal{A}} \varrho \mathbf{r} dA \times \mathbf{v}_G + \int_{\mathcal{A}} \varrho \mathbf{r} \times \left(\boldsymbol{\omega} \times \mathbf{r}\right) dA \end{aligned} \qquad [2.122]$$

where equation [2.69] has been used and $\mathbf{x}_Q = \mathbf{x}_G + \mathbf{r}$ has been posed. Due to the geometrical properties of G, the second and third integrals (whose integrands are linear in $\mathbf{r}$) vanish; hence:

$$\mathbf{h} = \mathbf{x} \times m\mathbf{v} + \int_{\mathcal{A}} \varrho \left[\left(\mathbf{r} \cdot \mathbf{r}\right) \boldsymbol{\omega} - \left(\mathbf{r} \cdot \boldsymbol{\omega}\right) \mathbf{r}\right] dA \qquad [2.123]$$

where $\mathbf{x} := \mathbf{x}_G$ has been posed and the double cross-product has been expanded[33]. By using the definition [2.117b] for the inertia tensor, equation [2.120] follows.

The stresses and the "Action and Reaction Principle"

We introduce magnitudes able to describe the mechanical interactions occurring among the body-points of the beam. Similar to what was done for the Cauchy continuum, we assume that the interaction only occurs through contact. Differently to

33. We remember the identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

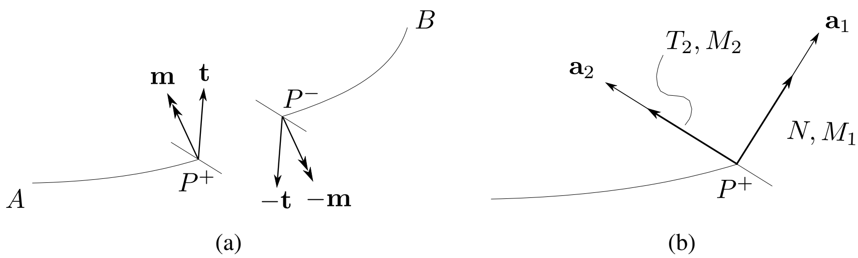


Figure 2.10: Force-stress and couple-stress: (a) Newton's third law, (b) scalar components.

that model, however, and as a peculiarity of the polar continuum, we admit that the body-points are able to exchange *contact internal couples* in addition to *contact internal forces*. This aspect should be considered as the dynamic counterpart of the ability to rotate possessed by the body-points. The internal actions are called *stresses*, in a generalized meaning, although they have the physical dimensions of forces (i.e. $[\mathrm{MLT}^{-2}]$) or couples ($[\mathrm{ML}^2\mathrm{T}^{-2}]$), respectively. The force should be interpreted as the integral of the (properly said) stresses acting on the cross-section of the beam, and the couple as the integral of the moment of these stresses with respect to the cross-section centroid. Thus, via the polar model, and as already observed in kinematics, we regain the information lost in passing from the 3D to 1D object.

With these ideas in mind, we define two vector fields: (a) the *force-stress* $\mathbf{t}(s,t)$, as the force exchanged at the abscissa s and time t between two points in contact, and (b) the *couple-stress* $\mathbf{m}(s,t)$, as the couple exchanged by the same points. We admit that the Newton's third law of action and reaction is satisfied, namely[34]:

$$\mathbf{t}^- = -\mathbf{t}^+, \qquad \mathbf{m}^- = -\mathbf{m}^+ \qquad [2.124]$$

in which the $\pm$ superscript denotes the positive, P^+, or the negative, P^-, "face" of the point P, where the stress is applied (see Figure 2.10(a)).

Conventionally, we will assume as positive the face that is on the boundary of a "volume" whose outward normal is concordant with $\mathbf{a}_1$, and we will simply denote by $\mathbf{t}, \mathbf{m}$ the stresses acting on it; the stresses acting on the opposite face will be denoted by $-\mathbf{t}, -\mathbf{m}$.

REMARK 2.13. It should be noted, that the Action and Reaction Principle is unimportant in the power balance approach, while, as we will see soon, it is essential in the force balance approach.

34. This is equivalent to the Cauchy Lemma of Continuum Mechanics.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The linear momentum balance equation

To write the momentum principles, we start by considering a finite portion of the beam, of end-points P_1^- and P_2^+, in its current configuration (Figure 2.11(a)). On this piece of beam, there are applied *external* forces $\mathbf{p}(s,t)$ and external couples $\mathbf{c}(s,t)$. Moreover, *internal* forces $-\mathbf{t}_1$, $\mathbf{t}_2$, and couples $-\mathbf{m}_1$, $\mathbf{m}_2$, are applied at the end-points, where $\mathbf{t}_i := \mathbf{t}(s_i,t), \mathbf{m}_i := \mathbf{m}(s_i,t)$ and equation [2.124] has been exploited.

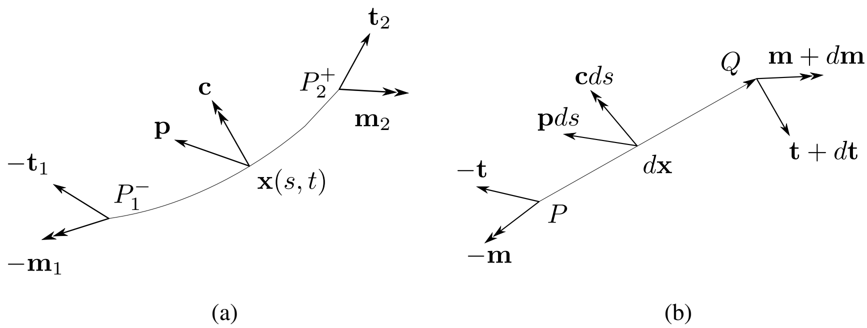


Figure 2.11: Internal and external forces acting on: (a) a finite portion of a beam, (b) an infinitesimal element.

The momentum principle requires that:

$$-\mathbf{t}_1 + \mathbf{t}_2 + \int_{s_1}^{s_2} \mathbf{p} ds = \partial_t \int_{s_1}^{s_2} m\mathbf{v} ds \qquad [2.125]$$

where m is the mass per unit length of the beam, assumed uniform, and the last term is the momentum time-rate (equation [2.119]). Note that the external forces act in the deformed configuration, but they refer to the original arc length; similarly for the mass [35]. Equation [2.125] can be transformed by bringing all terms under the integral sign[36]:

$$\int_{s_1}^{s_2} (\mathbf{t}' + \mathbf{p} - m\ddot{\mathbf{u}})\, ds = \mathbf{0} \qquad [2.126]$$

35. In an Eulerian approach, we would refer the principle to the current configuration by writing $\hat{\mathbf{p}}d\hat{s}$ and $\hat{m}d\hat{s}$ where a hat denotes a stretched quantity; however, continuity of force and mass entails that $\hat{\mathbf{p}}d\hat{s} = \mathbf{p}ds$ and $\hat{m}d\hat{s} = mds$.

36. This operation is the one-dimensional counterpart of the Green Lemma that permits us to transform a surface integral into a volume integral.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where $\mathbf{v} = \dot{\mathbf{u}}$ has been used. Since this integral must be equal to zero for any interval (s_1, s_2), the integrand must be zero everywhere, i.e.:

$$\mathbf{t}' + \mathbf{p} = m\ddot{\mathbf{u}} \qquad [2.127]$$

which is the *momentum equation* sought for (identical to equation [2.107a]). If dynamics effects can be neglected, it reduces to the equilibrium equation [2.103a].

The angular momentum balance equation

By selecting an arbitrary pole O, and taking, for simplicity $\mathbf{x}_O = \mathbf{0}$, the moment of all forces, internal and external, with respect to this point can be evaluated. The angular momentum principles states that:

$$\begin{aligned} &-\mathbf{m}_1 - \mathbf{x}_1 \times \mathbf{t}_1 + \mathbf{m}_2 + \mathbf{x}_2 \times \mathbf{t}_2 + \int_{s_1}^{s_2} (\mathbf{x} \times \mathbf{p} + \mathbf{c})\, ds \\ &\qquad = \partial_t \int_{s_1}^{s_2} (m\mathbf{x} \times \mathbf{v} + \mathbf{I}_G \boldsymbol{\omega})\, ds \end{aligned} \qquad [2.128]$$

where $\mathbf{x}_i := \mathbf{x}(s_i, t)$ and equation [2.120] has been accounted for by the angular moment. Here, the mass inertia tensor of the section with respect to the centroid, $\mathbf{I}_G$, is assumed to be diagonal and uniform along the beam[37]. By transforming the first terms in integral form, and accounting for $\partial_t \mathbf{x} \times \mathbf{v} = \mathbf{0}$, we have:

$$\int_{s_1}^{s_2} \left[\mathbf{m}' + (\mathbf{x} \times \mathbf{t})' + \mathbf{x} \times (\mathbf{p} - m\ddot{\mathbf{x}}) + (\mathbf{c} - \mathbf{I}_G \dot{\boldsymbol{\omega}})\right] ds = \mathbf{0} \qquad [2.129]$$

But, if equation [2.127] is used, this equation further reduces to:

$$\int_{s_1}^{s_2} [\mathbf{m}' + \mathbf{x}' \times \mathbf{t} + \mathbf{c} - \mathbf{I}_G \dot{\boldsymbol{\omega}}]\, ds = \mathbf{0} \qquad [2.130]$$

hence:

$$\mathbf{m}' + \mathbf{x}' \times \mathbf{t} + \mathbf{c} = \mathbf{I}_G \dot{\boldsymbol{\omega}} \qquad [2.131]$$

which is the *angular momentum equation* we looked for (identical to equation [2.107b]. If inertia effects are negligible, then this reduces to the equilibrium equation [2.103b].

37. Note that $\mathbf{I}_G$ is independent of time, since it represents the inertia of the cross-section with respect to the attached basis $\mathcal{B}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

REMARK 2.14. The equilibrium equations [2.103] can also be obtained by writing the balance of forces on an infinitesimal piece of beams, PQ, of length ds, namely (Figure 2.11(b)):

$$\begin{aligned} &-\mathbf{t}+\mathbf{t}+d\mathbf{t}+\mathbf{p}ds=\mathbf{0} \\ &-\mathbf{m}+\mathbf{m}+d\mathbf{m}+d\mathbf{x}\times(\mathbf{t}+d\mathbf{t})\,ds+\frac{1}{2}d\mathbf{x}\times\mathbf{p}ds+\mathbf{c}ds=\mathbf{0} \end{aligned} \qquad [2.132]$$

where $O \equiv P$ has been taken to simplify the expressions. By retaining the first-order quantities only, and dividing by ds, the equations previously obtained are reobtained. By invoking the d'Alembert principle, inertia forces $-m\ddot{\mathbf{u}}$ and inertia couples $-\mathbf{I}_G\dot{\boldsymbol{\omega}}$ must be added to obtain equations [2.127] and [2.131].

REMARK 2.15. Note that the balance equations, in vector form, are similar but *not* equal to the relevant expressions of the linear theory. Indeed, the arm of the force-stress in equation [2.131] is $\mathbf{x}' = \bar{\mathbf{a}}_1 + \mathbf{u}'$, while it is $\bar{\mathbf{a}}_1$ in the linear theory. Moreover, the normal and shear forces now act along the vectors of the current basis $\mathcal{B}$, while in the linear theory they are directed as the vectors of the reference basis $\bar{\mathcal{B}}$.

The mechanical boundary conditions

The *mechanical boundary conditions,* in the framework of the force balance approach, must be established separately from the field equations. These conditions prescribe the equality of the stresses "emerging" at the ends of the beam to the external forces applied there. Since the stresses acting at A^- are $-\mathbf{t}_A, -\mathbf{m}_A$, and those at B^+ are $\mathbf{t}_B, \mathbf{m}_B$, the conditions become (see Figure 2.9):

$$\mp\mathbf{t}_H = \mathbf{P}_H, \qquad \mp\mathbf{m}_H = \mathbf{C}_H, \qquad H = A, B \qquad [2.133]$$

where the upper (or lower) sign holds at A (or B).[38]

However, only *active* forces applied at the *free boundary* are known *a priori*; in contrast, *reactive* forces acting at the *constrained* boundary are unknown. Therefore, the mechanical boundary conditions [2.133] can only be prescribed at the free boundary. Thus, the same result [2.104] supplied by the virtual power formulation is obtained, i.e. natural and mechanical boundary conditions coincide.

38. Note that the boundary conditions [2.133] have a structure which is similar to that of the Cauchy continuum, i.e. $\mathbf{Tn} = \mathbf{f}$, where $\mathbf{T}$ is the stress tensor, $\mathbf{n}$ is the *outward* normal and $\mathbf{f}$ is the surface force. Here, in a 1D continuum, the stress tensor is replaced by a vector, $\mathbf{t}$ or $\mathbf{m}$, and the normal vector by a scalar, namely +1 on the right, where the outward normal is concordant with $\mathbf{a}_1$, and -1 on the left, where the outward normal is discordant with $\mathbf{a}_1$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2.2.4 The scalar forms of the balance equations and boundary conditions

The balance equations [2.107] and the mechanical (or natural) boundary conditions [2.133] can indifferently be projected onto the basis $\mathcal{B}$ or $\bar{\mathcal{B}}$. We will derive both forms of these scalar equations. In the next section, we will derive a third form.

The balance equations in the current basis

The space-derivatives of the vectors $\mathbf{t}$ and $\mathbf{m}$, being vectors attached to $\mathcal{B}$, follow the rule in equation [2.49]; the same occurs for the angular acceleration $\dot{\boldsymbol{\omega}}$ that must be evaluated by equation [2.64]. Therefore[39]:

$$\begin{aligned}
[\mathbf{t}']_{\mathcal{B}} &= \left[\sum_{i=1}^{3} T_i' \mathbf{a}_i + \mathbf{k}_c \times \mathbf{t}\right]_{\mathcal{B}} = \boldsymbol{t}' + \boldsymbol{K}\boldsymbol{t} \\
[\mathbf{m}']_{\mathcal{B}} &= \left[\sum_{i=1}^{3} M_i' \mathbf{a}_i + \mathbf{k}_c \times \mathbf{m}\right]_{\mathcal{B}} = \boldsymbol{m}' + \boldsymbol{K}\boldsymbol{m} \qquad [2.134] \\
[\dot{\boldsymbol{\omega}}]_{\mathcal{B}} &= \left[\sum_{i=1}^{3} \dot{\omega}_i \mathbf{a}_i + \boldsymbol{\omega} \times \boldsymbol{\omega}\right]_{\mathcal{B}} = \dot{\boldsymbol{\omega}}
\end{aligned}$$

In contrast, since the displacement $\mathbf{u} = \sum_{i=1}^{3} u_i \bar{\mathbf{a}}_i$ is represented in $\bar{\mathcal{B}}$, the linear acceleration is $\ddot{\mathbf{u}} = \sum_{i=1}^{3} \ddot{u}_i \bar{\mathbf{a}}_i$, and therefore[40]:

$$[\ddot{\mathbf{u}}]_{\mathcal{B}} = \left[\mathbf{R}^T \sum_{i=1}^{3} \ddot{u}_i \mathbf{a}_i\right]_{\mathcal{B}} = \boldsymbol{R}^T \ddot{\boldsymbol{u}} \qquad [2.135]$$

Concerning external forces, and depending on our convenience[41], we can represent them in either bases, e.g. $\mathbf{p} = \sum_{i=1}^{3} p_i \mathbf{a}_i$ or $\mathbf{p} = \sum_{i=1}^{3} \bar{p}_i \bar{\mathbf{a}}_i$, independent of the fact that we decided to project the equations in the current basis. If we use the former representation, we write $[\mathbf{p}]_{\mathcal{B}} = \boldsymbol{p}$, with $\boldsymbol{p} := (p_1, p_2, p_3)^T$; if we use the latter representation (as we do for the inertial forces), we write $[\mathbf{p}]_{\mathcal{B}} = \boldsymbol{R}^T \bar{\boldsymbol{p}}$, with

39. Here, $T_1 \equiv N$.

40. If, however, $\mathbf{u}$ is represented in $\mathcal{B}$, as for example could be useful for *rotating shafts*, then equation [2.65] must be used, leading to local, gyroscopic, centripetal and tangential components.

41. For example, dead loads are more conveniently expressed in the reference basis, while pressure or tangential forces are more conveniently expressed in the current basis.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$\bar{\boldsymbol{p}} := (\bar{p}_1, \bar{p}_2, \bar{p}_3)^T$. Similarly, for the other forces $\mathbf{c}$, $\mathbf{P}_H, \mathbf{C}_H$. Here, we use the representation in $\mathcal{B}$.

Therefore, the linear momentum equation, in the current basis, becomes:

$$\boldsymbol{t}' + \boldsymbol{K}\boldsymbol{t} + \boldsymbol{p} = m\boldsymbol{R}^T\ddot{\boldsymbol{u}} \quad [2.136]$$

or in extended form:

$$\begin{pmatrix} N' \\ T_2' \\ T_3' \end{pmatrix} + \begin{pmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{pmatrix} \begin{pmatrix} N \\ T_2 \\ T_3 \end{pmatrix} + \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}^T \begin{pmatrix} m\ddot{u}_1 \\ m\ddot{u}_2 \\ m\ddot{u}_3 \end{pmatrix} \quad [2.137]$$

By similar arguments, and remembering the definition [2.85], the scalar form in $\mathcal{B}$ of the angular momentum equation is[42]:

$$\boldsymbol{m}' + \boldsymbol{K}\boldsymbol{m} + \boldsymbol{\Lambda}\boldsymbol{t} + \boldsymbol{c} = \boldsymbol{I}_G\dot{\boldsymbol{\omega}} \quad [2.138]$$

or:

$$\begin{pmatrix} M_1' \\ M_2' \\ M_3' \end{pmatrix} + \begin{pmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} + \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -(1+\varepsilon) \\ -\gamma_2 & 1+\varepsilon & 0 \end{pmatrix} \begin{pmatrix} N \\ T_2 \\ T_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{pmatrix} \quad [2.139]$$

With regard to the mechanical boundary conditions, when equations [2.133] are projected onto the current basis, they furnish:

$$\mp\boldsymbol{t}_H = \boldsymbol{P}_H, \qquad \mp\boldsymbol{m}_H = \boldsymbol{C}_H \quad [2.140]$$

where $\boldsymbol{P}_H := (P_{1H}, P_{2H}, P_{3H})^T$, $\boldsymbol{C}_H := (C_{1H}, C_{2H}, C_{3H})^T$ are the components in $\mathcal{B}$ of forces and couples acting at the beam ends.

42. The rotatory inertia can also been expressed in terms of Tait–Bryan angles by using equation [2.71b].

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The balance equations in the reference basis

To express the balance equations in $\bar{\mathcal{B}}$, we can use the scalar representation obtained in $\mathcal{B}$ and perform a change of basis. Accordingly, we have to pre-multiply the column matrix that collects the equations by the rotation matrix $\boldsymbol{R} =: [R_{ij}]$. Thus, equations [2.136] and [2.138] transform as follows[43]:

$$\begin{aligned} &\boldsymbol{R}\left(\boldsymbol{t}' + \boldsymbol{K}\boldsymbol{t}\right) + \bar{\boldsymbol{p}} = m\ddot{\boldsymbol{u}} \\ &\boldsymbol{R}\left(\boldsymbol{m}' + \boldsymbol{K}\boldsymbol{m}\right) + \boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{t} + \bar{\boldsymbol{c}} = \boldsymbol{R}\boldsymbol{I}_G\dot{\boldsymbol{\omega}} \end{aligned} \qquad [2.141]$$

where we used components of forces in $\bar{\mathcal{B}}$.

Concerning the mechanical boundary conditions, when equations [2.133] are projected onto the reference basis, we have:

$$\mp \mathbf{R}_H \boldsymbol{t}_H = \bar{\boldsymbol{P}}_H, \qquad \mp \mathbf{R}_H \boldsymbol{m}_H = \bar{\boldsymbol{C}}_H \qquad [2.142]$$

where $\bar{\boldsymbol{P}}_H := \left(\bar{P}_{1H}, \bar{P}_{2H}, \bar{P}_{3H}\right)^T$, $\bar{\boldsymbol{C}}_H := \left(\bar{C}_{1H}, \bar{C}_{2H}, \bar{C}_{3H}\right)^T$ list the component of forces and couples acting at $H = A, B$ in the basis $\bar{\mathcal{B}}$.

REMARK 2.16. The translational inertia forces assume a simpler form in reference basis, while the angular inertia forces are simpler in the current basis.

2.2.5 The Lagrangian balance equations

In the previous sections, we derived the vector form of the balance equations, and then projected it onto a basis to our liking. There, all magnitudes involved have a vector (or tensor) character. However, as is well-known, there exists an alternative approach, in which magnitudes are handled as scalar quantities. As an example, the Euler–Lagrange equations of motions of a finite-dimensional system involve the Lagrangian coordinates, which are not components of a vector, but a collection of scalar quantities.

In our treatment, we indeed encountered a similar collection, namely the pseudo-vector (by Argyris [ARG 82]) $\boldsymbol{\theta}$, which lists the Tait–Bryan angles θ_i. Their time-derivatives $\dot{\theta}_i$ naturally appear when strain–rates are evaluated by time-differentiation of the strain-displacement relationships, instead of using their

43. Collecting stresses and forces in column-vectors, i.e. $\boldsymbol{\sigma} := (N, T_2, T_3, M_1, M_2, M_3)^T$, $\boldsymbol{p} := (p_1, p_2, p_3, c_1, c_2, c_3)^T$, they become $\boldsymbol{D}^\star\left(\boldsymbol{w}, \boldsymbol{w}'\right)\boldsymbol{\sigma} = \boldsymbol{p}$. Here, $\boldsymbol{D}^\star\left(\boldsymbol{w}, \boldsymbol{w}'\right)$ is the 6×6 *equilibrium operator*, depending on the six configuration variables $\boldsymbol{w}$ and their derivatives (via the rotations θ_i and the strain–displacement relationships). The equilibrium operator $\boldsymbol{D}^\star\left(\boldsymbol{w}, \boldsymbol{w}'\right)$ is the *adjoint* of the kinematic operator $\boldsymbol{D}\left(\boldsymbol{w}, \boldsymbol{w}'\right)$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

relations with the stretching and curvature gradient. Therefore, it is expected that if the pseudo-vector is used, scalar balance equations, different from those derived above, are found.

We want to investigate this topic with a twofold scope: (a) to understand the link existing among the different forms of the scalar balance equations and (b) to prepare the ground for further developments (namely, formulation of internally constrained models, Chapter 4, where the scalar approach is more useful).

The scalar balance equations in the reference basis

We start by rewriting the VPP in scalar form, namely:

$$\int_{\mathcal{S}} \left(\dot{\boldsymbol{e}}^T\boldsymbol{t} + \dot{\boldsymbol{k}}^T\boldsymbol{m}\right) ds = \int_{\mathcal{S}} \left(\boldsymbol{v}^T\bar{\boldsymbol{p}} + \bar{\boldsymbol{\omega}}^T\bar{\boldsymbol{c}}\right) ds + \sum_{H=A}^{B} \left[\boldsymbol{v}^T\bar{\boldsymbol{P}} + \bar{\boldsymbol{\omega}}^T\bar{\boldsymbol{C}}\right]_H \quad [2.143]$$

where all the quantities are evaluated in the reference basis. Here, the internal virtual power appears as the product of column matrices, namely the strain-rates $\dot{\boldsymbol{e}} := (\dot{\varepsilon}, \dot{\gamma}_2, \dot{\gamma}_3)^T$, $\dot{\boldsymbol{k}} := (\dot{\kappa}_1, \dot{\kappa}_2, \dot{\kappa}_3)^T$, we introduced in section 2.1.8, and the dual stresses $\boldsymbol{t} := (N, T_2, T_3)^T$, $\boldsymbol{m} := (M_1, M_2, M_3)^T$. By expressing (by the way of equations [2.87]) the strain-rates in terms of velocity and spin components, i.e. $\boldsymbol{v} := (\dot{u}_1, \dot{u}_2, \dot{u}_3)$, $\bar{\boldsymbol{\omega}} := (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)$, and integrating by parts, the internal virtual power becomes:

$$\begin{aligned}\mathcal{P}_{int} &:= \int_{\mathcal{S}} \left(\boldsymbol{v}'^T\boldsymbol{R}\boldsymbol{t} + \bar{\boldsymbol{\omega}}^T\boldsymbol{R}\boldsymbol{\Lambda}^T\boldsymbol{t} + \bar{\boldsymbol{\omega}}'^T\boldsymbol{R}\boldsymbol{m}\right) ds \\ &= \int_{\mathcal{S}} \left[-\boldsymbol{v}^T(\boldsymbol{R}\boldsymbol{t})' - \bar{\boldsymbol{\omega}}^T\boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{t} - \bar{\boldsymbol{\omega}}^T(\boldsymbol{R}\boldsymbol{m})'\right] ds \\ &\quad + \left[\boldsymbol{v}^T\boldsymbol{R}\boldsymbol{t} + \bar{\boldsymbol{\omega}}^T\boldsymbol{R}\boldsymbol{m}\right]_A^B\end{aligned} \quad [2.144]$$

where we used $\boldsymbol{\Lambda}^T = -\boldsymbol{\Lambda}$; therefore, the VPP becomes:

$$\begin{aligned}&\int_{\mathcal{S}} \left\{\boldsymbol{v}^T\left[(\boldsymbol{R}\boldsymbol{t})' + \bar{\boldsymbol{p}}\right] + \bar{\boldsymbol{\omega}}^T\left[(\boldsymbol{R}\boldsymbol{m})' + \boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{t} + \bar{\boldsymbol{c}}\right]\right\} ds \\ &\quad + \sum_{H=A}^{B} \left[\boldsymbol{v}^T\left(\bar{\boldsymbol{P}} \pm \boldsymbol{R}\boldsymbol{t}\right) + \bar{\boldsymbol{\omega}}^T\left(\bar{\boldsymbol{C}} \pm \boldsymbol{R}\boldsymbol{m}\right)\right]_H = 0 \qquad \forall\,(\boldsymbol{v}, \bar{\boldsymbol{\omega}})\end{aligned} \quad [2.145]$$

From this, the static version of the scalar balance equations [2.141] and the boundary conditions [2.140] follow once the derivative of the product has been performed and $\boldsymbol{K} = \boldsymbol{R}^T\boldsymbol{R}'$ has been taken into account.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The Lagrangian balance equations

According to the Lagrangian formulation, the configuration variables are the translations $\boldsymbol{u}$ and the elementary rotations $\boldsymbol{\theta}$, so that the virtual motion is described by the velocities $\boldsymbol{v} \equiv \dot{\boldsymbol{u}}$ and the time-derivatives $\dot{\boldsymbol{\theta}}$, instead of $\bar{\boldsymbol{\omega}}$. Since these latter quantities are related by $\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{B}}_\omega \dot{\boldsymbol{\theta}}$ (equation [2.71a]), the VPP [2.145] modifies into:

$$\int_{\mathcal{S}} \left\{ \dot{\boldsymbol{u}}^T \left[(\boldsymbol{Rt})' + \bar{\boldsymbol{p}} \right] + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \left[(\boldsymbol{Rm})' + \boldsymbol{R\Lambda t} + \bar{\boldsymbol{c}} \right] \right\} ds$$
$$+ \sum_{H=A}^{B} \left[\dot{\boldsymbol{u}}^T \left(\bar{\boldsymbol{P}} \pm \boldsymbol{Rt} \right) + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \left(\bar{\boldsymbol{C}} \pm \boldsymbol{Rm} \right) \right]_H = 0 \qquad \forall \left(\dot{\boldsymbol{u}}, \dot{\boldsymbol{\theta}} \right) \tag{2.146}$$

from which the balance equations follow[44]:

$$\begin{aligned} &\boldsymbol{R}\left(\boldsymbol{t}' + \boldsymbol{Kt}\right) + \bar{\boldsymbol{p}} = \boldsymbol{0} \\ &\bar{\boldsymbol{B}}_\omega^T \left[\boldsymbol{R}\left(\boldsymbol{m}' + \boldsymbol{Km}\right) + \boldsymbol{R\Lambda t} + \bar{\boldsymbol{c}} \right] = \boldsymbol{0} \end{aligned} \tag{2.147}$$

with the boundary conditions:

$$\begin{aligned} &\dot{\boldsymbol{u}}_H^T \left(\bar{\boldsymbol{P}}_H \pm \boldsymbol{R}_H \boldsymbol{t}_H \right) = \boldsymbol{0} \\ &\dot{\boldsymbol{\theta}}_H^T \bar{\boldsymbol{B}}_{\omega H}^T \left(\bar{\boldsymbol{C}}_H \pm \boldsymbol{R}_H \boldsymbol{m}_H \right) = \boldsymbol{0} \end{aligned} \tag{2.148}$$

These will be referred to as the *Lagrangian equations*. By comparing them with equations [2.141] and [2.140], it appears that while the linear momentum equations coincide, the angular momentum equations differ. As a matter of fact, by remembering that the columns of $\bar{\boldsymbol{B}}_\omega$ are the director cosines of the three elementary rotation axes with respect to the reference basis (i.e. the unit vectors of the spin basis $\mathcal{B}_\omega$), we conclude that *they are the projection of the moment vector equations on these axes*. Therefore, according to the Lagrangian formulations, while the translation equations represent the force balance along the orthogonal directions of $\bar{\mathcal{B}}$, the moment equations express the balance of moments around the non-orthogonal directions of $\mathcal{B}_\omega$.

REMARK 2.17. Note that $\bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}}$ and $\bar{\boldsymbol{B}}_{\omega H}^T \bar{\boldsymbol{C}}_H$ are just the *Lagrangian forces* of the Analytical Mechanics, i.e. the forces which spend power on the time-derivatives of the Lagrange parameters, $\dot{\boldsymbol{\theta}}$ and $\dot{\boldsymbol{\theta}}_H$, respectively.

44. When these equations are written in formal matrix form, as we did for the metamodel (section 1.2.2), they become:

$$-\left(\begin{bmatrix} \boldsymbol{RK} & \boldsymbol{0} \\ \bar{\boldsymbol{B}}_\omega^T \boldsymbol{R\Lambda} & \bar{\boldsymbol{B}}_\omega^T \boldsymbol{RK} \end{bmatrix} + \begin{bmatrix} \boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & \bar{\boldsymbol{B}}_\omega^T \boldsymbol{R} \end{bmatrix} \partial_s \right) \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{m} \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{p}} \\ \bar{\boldsymbol{c}} \end{pmatrix}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2.3 Constitutive law

The formulation of the model calls for introducing a *constitutive law* characterizing the material behavior, aimed at establishing a link among stresses and strains. We mainly confine ourselves to hyperelastic materials; however, we also give an outline for linear viscoelastic materials.

2.3.1 The hyperelastic law

As we discussed about the metamodel in section 1.2.3, a beam is hyperelastic when the work spent by the external forces to deform it is equal to the variation of an elastic energy, whose existence is postulated. This property excludes any internal dissipation. By referring to an interval of time dt, the work spent by the external forces acting on the beam in the *true* (not virtual) velocity field is $\mathcal{P}_{ext}dt$, which, by virtue of the VPP [2.99], is equal to $\mathcal{P}_{int}dt$. Therefore, the *deformation work for unit length* of the beam is;

$$\begin{aligned}\frac{d}{ds}\left(\mathcal{P}_{int}dt\right) &= \left(\mathbf{t}\cdot\mathbf{R}\dot{\mathbf{e}} + \mathbf{m}\cdot\mathbf{R}\dot{\mathbf{k}}\right)dt \\ &= \mathbf{t}\cdot\mathbf{R}d\mathbf{e} + \mathbf{m}\cdot\mathbf{R}d\mathbf{k} \\ &= d\mathbf{e}\cdot\mathbf{R}^T\mathbf{t} + d\mathbf{k}\cdot\mathbf{R}^T\mathbf{m}\end{aligned} \qquad [2.149]$$

where the expression [2.96] for $\mathcal{P}_{int}$ and an identity have been used. In order for this expression to be an *exact differential*, it must equate the differential of an *elastic potential* function $\phi(\mathbf{e},\mathbf{k})$, which, by definition, becomes:

$$d\phi = d\mathbf{e}\cdot\frac{\partial\phi(\mathbf{e},\mathbf{k})}{\partial\mathbf{e}} + d\mathbf{k}\cdot\frac{\partial\phi(\mathbf{e},\mathbf{k})}{\partial\mathbf{k}} \qquad [2.150]$$

By equating the previous expressions, the hyperelastic (or *Green*) law follows:

$$\mathbf{R}^T\mathbf{t} = \frac{\partial\phi(\mathbf{e},\mathbf{k})}{\partial\mathbf{e}}, \qquad \mathbf{R}^T\mathbf{m} = \frac{\partial\phi(\mathbf{e},\mathbf{k})}{\partial\mathbf{k}} \qquad [2.151]$$

It states that *the pulled-back stresses,* $\mathbf{R}^T\mathbf{t}$, $\mathbf{R}^T\mathbf{m}$*, equate the derivatives of the elastic potential with respect to the dual (reference) strains* $\mathbf{e},\mathbf{k}$*, respectively*. Therefore, $\mathbf{R}^T$ accounts for the fact that the stresses are defined in the current configuration, while the strains are defined in the reference configuration.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

In scalar form, since:

$$\begin{aligned}
\mathbf{R}^T\mathbf{t} &= N\bar{\mathbf{a}}_1 + T_2\bar{\mathbf{a}}_2 + T_3\bar{\mathbf{a}}_3 \\
\mathbf{R}^T\mathbf{m} &= M_1\bar{\mathbf{a}}_1 + M_2\bar{\mathbf{a}}_2 + M_3\bar{\mathbf{a}}_3 \\
\frac{\partial\phi}{\partial\mathbf{e}} &= \frac{\partial\phi}{\partial\varepsilon}\bar{\mathbf{a}}_1 + \frac{\partial\phi}{\partial\gamma_2}\bar{\mathbf{a}}_2 + \frac{\partial\phi}{\partial\gamma_3}\bar{\mathbf{a}}_3 \\
\frac{\partial\phi}{\partial\mathbf{k}} &= \frac{\partial\phi}{\partial\kappa_1}\bar{\mathbf{a}}_1 + \frac{\partial\phi}{\partial\kappa_2}\bar{\mathbf{a}}_2 + \frac{\partial\phi}{\partial\kappa_3}\bar{\mathbf{a}}_3
\end{aligned} \qquad [2.152]$$

the hyperelastic law becomes:

$$\begin{aligned}
N &= \frac{\partial\phi}{\partial\varepsilon}, \quad T_2 = \frac{\partial\phi}{\partial\gamma_2}, \quad T_3 = \frac{\partial\phi}{\partial\gamma_3} \\
M_1 &= \frac{\partial\phi}{\partial\kappa_1}, \quad M_2 = \frac{\partial\phi}{\partial\kappa_2}, \quad M_3 = \frac{\partial\phi}{\partial\kappa_3}
\end{aligned} \qquad [2.153]$$

with $\phi = \phi\left(\varepsilon, \gamma_2, \gamma_3, \kappa_1, \kappa_2, \kappa_3\right)$. In general, these laws are *nonlinear* and *coupled*.

The Hooke law

As an example of a widely used elastic law, let us assume that the potential is a complete quadratic form in the strains, namely:

$$\begin{aligned}
\phi(\mathbf{e},\mathbf{k}) :=& \phi_0 + \left(\mathbf{e}\cdot\mathring{\mathbf{t}} + \mathbf{k}\cdot\mathring{\mathbf{m}}\right) \\
&+ \frac{1}{2}\left(\mathbf{e}\cdot\mathbf{E}_{ee}\mathbf{e} + \mathbf{e}\cdot\mathbf{E}_{ek}\mathbf{k} + \mathbf{k}\cdot\mathbf{E}_{ke}\mathbf{e} + \mathbf{k}\cdot\mathbf{E}_{kk}\mathbf{k}\right)
\end{aligned} \qquad [2.154]$$

where ϕ_0 is an inessential constant, and $\mathbf{E}_{\alpha\beta} = \mathbf{E}_{\beta\alpha}^T$ are *elastic tensors*, whose symmetry properties came down to the fact that $\phi \equiv \phi^T$. Then, according to equations [2.151], the hyperelastic law is linear:

$$\begin{aligned}
\mathbf{R}^T\mathbf{t} &= \mathring{\mathbf{t}} + \mathbf{E}_{ee}\mathbf{e} + \mathbf{E}_{ek}\mathbf{k} \\
\mathbf{R}^T\mathbf{m} &= \mathring{\mathbf{m}} + \mathbf{E}_{ke}\mathbf{e} + \mathbf{E}_{kk}\mathbf{k}
\end{aligned} \qquad [2.155]$$

and it is known as the (non-homogeneous) *Hooke law*. In it, the *prestresses*, $\mathring{\mathbf{t}}$, $\mathring{\mathbf{m}}$ appear, i.e. possible stresses acting in the reference configuration, at which $\mathbf{e} = \mathbf{k} = \mathbf{0}$. If the natural configuration is taken as a reference, then prestresses vanish, and the Hooke law becomes homogeneous.

In matrix form (by equations [2.152a,b]), we can also write:

$$\begin{aligned}
\boldsymbol{t} &= \mathring{\boldsymbol{t}} + \boldsymbol{E}_{ee}\mathbf{e} + \boldsymbol{E}_{ek}\boldsymbol{k} \\
\boldsymbol{m} &= \mathring{\boldsymbol{m}} + \boldsymbol{E}_{ke}\mathbf{e} + \boldsymbol{E}_{kk}\boldsymbol{k}
\end{aligned} \qquad [2.156]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where all vectors and matrices collect components in the reference basis. In extended form, we have:

$$\begin{pmatrix} N \\ T_2 \\ T_3 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} \overset{\circ}{N} \\ \overset{\circ}{T}_2 \\ \overset{\circ}{T}_3 \\ \overset{\circ}{M}_1 \\ \overset{\circ}{M}_2 \\ \overset{\circ}{M}_3 \end{pmatrix} + \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ & & & & E_{55} & E_{56} \\ \text{sym} & & & & & E_{66} \end{pmatrix} \begin{pmatrix} \varepsilon \\ \gamma_2 \\ \gamma_3 \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{pmatrix} \quad [2.157]$$

in which the 6×6 elastic matrix $[E_{ij}]$ is full, depending on 21 independent constants.

2.3.2 Identification of the elastic law from a 3D-model

To formulate a hyperelastic law, we must, in this order: (a) adopt an elastic potential and (b) attribute a value to the elastic constants contained in it. Concerning task (a), the simplest choice consists of taking a polynomial expression, of second or higher degree. The higher the degree, the higher the possibility of fitting experimental results. Usually, a quadratic expression, leading to the Hooke law, is adopted in application, but we will show soon that this drastic simplification suffers some drawbacks. Concerning task (b), if experimental data are not available, one can perform an *identification* of the elastic constants from a 3D-model, for which the elastic potential is known. The procedure requires to express the strains of the 3D-model in terms of the strains of the 1D-model via a *strain-map*. Once this has been built up, the elastic potential for the 3D-model is assumed as the potential for the 1D-model. Here, the procedure is explained for a linear and a (simplified) nonlinear model.

The strain map

A 3D-beam is considered, which is made up of linear elastic and isotropic material. Its elastic potential for unit length becomes:

$$\phi = \int_{\mathcal{A}} \left(\overset{\circ}{\sigma}_{11}\varepsilon_{11} + \overset{\circ}{\tau}_{12}\gamma_{12} + \overset{\circ}{\tau}_{13}\gamma_{13}\right) dA + \frac{1}{2}\int_{\mathcal{A}} \left[E\varepsilon_{11}^2 + G\left(\gamma_{12}^2 + \gamma_{13}^2\right)\right] dA \quad [2.158]$$

where $\mathcal{A}$ is the cross-section area, E is the Young modulus, G is the tangential elastic modulus, ε_{11} is the unit extension of the longitudinal fibers, γ_{12}, γ_{13} are the shear strains and $\overset{\circ}{\sigma}_{11}, \overset{\circ}{\tau}_{12}, \overset{\circ}{\tau}_{13}$ are the pre-existing stresses. By assuming that the cross-sections of the beam remain planar and adopting *linear kinematics*, the strain

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

field on the cross-section becomes:

$$\begin{aligned} \varepsilon_{11} &= \varepsilon + \kappa_2 r_3 - \kappa_3 r_2 \\ \gamma_{12} &= \gamma_2 - \kappa_1 r_3 \\ \gamma_{13} &= \gamma_3 + \kappa_1 r_2 \end{aligned} \qquad [2.159]$$

It depends on six constants $(\varepsilon; \gamma_2, \gamma_3; \kappa_1, \kappa_2, \kappa_3)$, denoting, in order: the unit extension at the centroid, the uniform (averaged) shear strains and the torsional and flexural curvatures; moreover, $\overrightarrow{GQ} := \bar{\mathbf{r}} = r_2\bar{\mathbf{a}}_2 + r_3\bar{\mathbf{a}}_3$ selects the generic point Q with respect to the centroid G. If we identify the six constants using homonymous quantities of the 1D-model, equations [2.159] takes the role of a *strain-map*, linking the strain field of the 3D-model to the strains of the 1D-model.

A linear uncoupled law

By using equation [2.159] in equation [2.158], performing integrations and by assuming that $\bar{\mathcal{B}}$ is the principal inertia basis, a quadratic non-homogeneous expression for the potential is obtained:

$$\begin{aligned} \phi =& \mathring{N}\varepsilon + \mathring{T}_2\gamma_2 + \mathring{T}_3\gamma_3 + \mathring{M}_1\kappa_1 + \mathring{M}_2\kappa_2 + \mathring{M}_3\kappa_3 \\ &+ \frac{1}{2}\left(EA\varepsilon^2 + GA\gamma_2^2 + GA\gamma_3^2 + GJ_G\kappa_1^2 + EJ_2\kappa_2^2 + EJ_3\kappa_3^2\right) \end{aligned} \qquad [2.160]$$

where, A is the cross-section area, J_G is the area polar moment of inertia with respect to the centroid and J_2, J_3 are the area moments of inertia with respect to the two principal axes, i.e.:

$$A := \int_{\mathcal{A}} dA, \qquad J_2 = \int_{\mathcal{A}} r_3^2 dA, \qquad J_3 = \int_{\mathcal{A}} r_2^2 dA, \qquad J_G = \int_{\mathcal{A}} r^2 dA \quad [2.161]$$

where $r := \|\bar{\mathbf{r}}\|$; moreover:

$$\begin{aligned} \mathring{N} &:= \int_{\mathcal{A}} \mathring{\sigma}_{11} dA, \quad \mathring{M}_1 := \int_{\mathcal{A}} (\mathring{\tau}_{13} r_2 - \mathring{\tau}_{12} r_3)\, dA \\ \mathring{T}_2 &:= \int_{\mathcal{A}} \mathring{\tau}_{12} dA, \quad \mathring{M}_2 := \int_{\mathcal{A}} \mathring{\sigma}_{11} r_3 dA \\ \mathring{T}_3 &:= \int_{\mathcal{A}} \mathring{\tau}_{13} dA, \quad \mathring{M}_3 := -\int_{\mathcal{A}} \mathring{\sigma}_{11} r_2 dA \end{aligned} \qquad [2.162]$$

are the prestresses for the 1D-model.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

However, as is well-known, this expression is quite inaccurate in describing the contribution of shear strains and torsion because we neglected the relevant warping. Therefore, although in an inelegant way, the potential is usually "corrected" into:

$$\begin{aligned}\tilde{\phi} =& \overset{\circ}{N}\varepsilon + \overset{\circ}{T}_2\gamma_2 + \overset{\circ}{T}_3\gamma_3 + \overset{\circ}{M}_1\kappa_1 + \overset{\circ}{M}_2\kappa_2 + \overset{\circ}{M}_3\kappa_3 \\ &+ \frac{1}{2}\left(EA\varepsilon^2 + GA_2\gamma_2^2 + GA_3\gamma_3^2 + GJ_1\kappa_1^2 + EJ_2\kappa_2^2 + EJ_3\kappa_3^2\right)\end{aligned} \quad [2.163]$$

where A_2, A_3 are *shear-areas* and J_1 is the torsional moment of inertia[45]. Here, EA is the *axial stiffness*, GA_j is the *shear stiffnesses*, GJ_1 is the *torsional stiffness* and EJ_j is the *bending stiffnesses* $(j = 2, 3)$. In the 1D-model, each double symbol should be considered as a "whole symbol", including both the material properties and the geometric properties of the (disappeared) cross-section.

The potential [2.163] is a special case of the more general form [2.154], in which no mixed terms in the strains appear. Therefore, in the symbolism of equation [2.157], *the elastic matrix* $[E_{ij}]$ *is diagonal*. The uncoupled constitutive law, hence, becomes:

$$\begin{aligned}&N = \overset{\circ}{N} + EA\varepsilon, \qquad T_2 = \overset{\circ}{T}_2 + GA_2\gamma_2, \qquad T_3 = \overset{\circ}{T}_3 + GA_3\gamma_3 \\ &M_1 = \overset{\circ}{M}_1 + GJ_1\kappa_1, \quad M_2 = \overset{\circ}{M}_2 + EJ_2\kappa_2, \quad M_3 = \overset{\circ}{M}_3 + EJ_3\kappa_3\end{aligned} \quad [2.164]$$

A nonlinear constitutive law for large twist

The linear, uncoupled law [2.164] is widely adopted in applications. However, it suffers a serious setback when the beam undergoes large twist. To bring the problem to light, let us consider a beam in pure torsion $(u_i \equiv 0, \theta_1 \neq 0, \theta_2 = \theta_3 \equiv 0)$. From the strain–displacement relationships [2.52] and [2.53], it follows that the unique not vanishing strain is $\kappa_1 = \theta_1'$, which is a linear relation. Moreover, since $\mathbf{u} = \mathbf{0}$, the equilibrium equations [2.103] also become linear; in particular, $M_1' + c_1 = 0$. If, therefore, we also adopt a linear constitutive law, such as $M_1 = GJ_1\kappa_1$, *the whole problem is linear, irrespective of the amplitude of the twist.* Since this result is unrealistic, and kinematics and equilibrium are exact, we must conclude that the inadequacy of the model lies in the constitutive law, which must be taken as nonlinear. On a physical ground, indeed, we have to take into account that, when the 3D-beam is twisted, the longitudinal fibers elongate in passing from straight lines to helicoidal spirals.

Although inconsistently, we improve the model by adding just a term to equation [2.159a], in which we include the effect of twist. Accordingly, the unit extension

45. The shear areas are evaluated resorting to the exact theory of non-uniform bending, or to the simplified Jourawsky theory, while the torsional inertia is computed via the torsion theory, stated in the Neumann or Dirichlet forms.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

becomes:

$$\varepsilon_{11} = \sqrt{(1 + u_1')^2 + \|\mathbf{u}_\pi'\|^2} - 1 \qquad [2.165]$$

where we wrote $\mathbf{u} = u_1\bar{\mathbf{a}}_1 + \mathbf{u}_\pi$, with $\mathbf{u}_\pi$ the in-plane component. Since, in linear kinematics, $\mathbf{u}_\pi = \theta_1\bar{\mathbf{a}}_1 \times \bar{\mathbf{r}}$, then $\|\mathbf{u}_\pi'\| = \kappa_1 r$; hence, by expanding in series:

$$\varepsilon_{11} = u_1' + \frac{1}{2}\kappa_1^2 r^2 + \text{higher order terms} \qquad [2.166]$$

Equation [2.159a] is consequently updated as:

$$\varepsilon_{11} = \varepsilon + \kappa_2 r_3 - \kappa_3 r_2 + \frac{1}{2}\kappa_1^2 r^2 \qquad [2.167]$$

while equations [2.159b,c] are left unchanged. Therefore, the potential [2.158] modifies because of the appearance of the extra-terms:

$$\begin{aligned} \Delta\phi :=& \frac{1}{2}\kappa_1^2 \int_{\mathcal{A}} \mathring{\sigma}_{11} r^2 dA \\ &+ \frac{1}{2}\left(EJ_G\varepsilon\kappa_1^2 - EY_{3G}\kappa_3\kappa_1^2 + EY_{2G}\kappa_2\kappa_1^2\right) + \text{h.o.t.} \end{aligned} \qquad [2.168]$$

where only prestress and cubic terms have been retained. Here, J_G is the already-encountered, area polar inertia moment, and new, third-order area moments, of dimensions $\left[\mathrm{L}^5\right]$, have been introduced[46]:

$$Y_{2G} := \int_{\mathcal{A}} r_3 r^2 dA, \qquad Y_{3G} := \int_{\mathcal{A}} r_2 r^2 dA \qquad [2.169]$$

Note that $Y_{2G} = 0$ if $\bar{\mathbf{a}}_2$ is a symmetry axis for the section; similarly, $Y_{3G} = 0$ if $\bar{\mathbf{a}}_3$ is of symmetry. The prestress contribution, moreover, can be transformed assuming that (Navier formula):

$$\mathring{\sigma}_{11} = \frac{\mathring{N}}{A} + \frac{\mathring{M}_2}{J_2} r_3 - \frac{\mathring{M}_3}{J_3} r_2 \qquad [2.170]$$

so that:

$$\int_{\mathcal{A}} \mathring{\sigma}_{11} r^2 dA = \mathring{N}\rho_G^2 + \mathring{M}_2 \frac{Y_{2G}}{J_2} - \mathring{M}_3 \frac{Y_{3G}}{J_3} \qquad [2.171]$$

46. As a general notation adopted in this book, $[J_\alpha] = \left[\mathrm{L}^4\right]$, $[Y_\alpha] = \left[\mathrm{L}^5\right]$ and, to be used later, $[\Gamma_\alpha] = \left[\mathrm{L}^6\right]$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where $\rho_G^2 := J_G/A$ is the polar radius of inertia with respect to the centroid.

By differentiating the extended potential $\tilde{\phi} + \Delta\phi$, the following quadratic constitutive law is obtained:

$$\begin{aligned} N &= \mathring{N} + EA\varepsilon + \frac{1}{2}EJ_G\kappa_1^2 \\ T_2 &= \mathring{T}_2 + GA_2\gamma_2 \\ T_3 &= \mathring{T}_3 + GA_3\gamma_3 \\ M_1 &= \mathring{M}_1 + GJ_1^\star\kappa_1 + EJ_G\varepsilon\kappa_1 - EY_{3G}\kappa_3\kappa_1 + EY_{2G}\kappa_2\kappa_1 \\ M_2 &= \mathring{M}_2 + EJ_2\kappa_2 + \frac{1}{2}EY_{2G}\kappa_1^2 \\ M_3 &= \mathring{M}_3 + EJ_3\kappa_3 - \frac{1}{2}EY_{3G}\kappa_1^2 \end{aligned} \qquad [2.172]$$

where:

$$GJ_1^\star := GJ_1 + \mathring{N}\rho_G^2 + \mathring{M}_2\frac{Y_{2G}}{J_2} - \mathring{M}_3\frac{Y_{3G}}{J_3} \qquad [2.173]$$

It appears that the quadratic nonlinearities couple torsion to extension and bending. Thus, if a double-symmetric beam is twisted, axial forces must be applied at the ends in order to keep the length unchanged; otherwise, a *shortening* occurs. If the cross-section is not symmetric, couples are also necessary, otherwise the beam bends itself. Moreover, if the beam is prestressed by an axial force and/or bending moment, these stresses contribute to the torsional moment. This is an effect which is well-known in elastic stability of beams (see e.g. [PIG 92]), where axial forces and bending moments can produce torsional/lateral buckling.

REMARK 2.18. At a first glance, it could appear that the part of the elastic energy [2.168] which is independent of the prestress is not positive-definite. However, it must be kept in mind that this expression is truncated, and therefore it is invalid when third-order terms become of the same order as second-order terms. If a non-truncated expression were instead adopted, the contribution to the elastic energy would be positive-definite, since it comes down to the integral of the square of ε_{11}. However, we must not forget that ε_{11} itself (equation [2.167]) is approximated, since it does not vanish for any rigid motions. Therefore, the constitutive law [2.172] should be used in a perturbation perspective only.

REMARK 2.19. Equation [2.173] can be explained as follows. When the beam is prestressed, because of the fact that the longitudinal fibers incline with respect to the (assumed planar) cross-section, the pre-existing stresses also incline, so that they

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

have a non-zero moment with respect to the beam axis. A simple experiment highlighting this effect consists of twisting a two-string swing; since the strings, which are prestressed by the plate's own weight, are no longer vertical in the current configuration, an apparent (geometric) torsional stiffness manifests itself, giving rise to twist oscillations, although no first-order elastic strains occur.

Rotation about the center of twist

A more refined analysis of the twist requires accounting for twist rotations occurring not around the centroid, but around the center of twist (or flexural center or shear center). The question is not really relevant for compact cross-sections, but it is very important for thin-walled beams (see the later Chapter 7). In the context of the locally undeformable beams we are dealing with, we can easily include this aspect in the analysis, by rewriting equations [2.167] and [2.159b,c] as:

$$\begin{aligned} \varepsilon_{11} &= \varepsilon + \kappa_2 r_3 - \kappa_3 r_2 + \frac{1}{2}\kappa_1^2 \left(r - r_C\right)^2 \\ \gamma_{12} &= \gamma_2 - \kappa_1 \left(r_3 - r_{3C}\right) \\ \gamma_{13} &= \gamma_3 + \kappa_1 \left(r_2 - r_{2C}\right) \end{aligned} \qquad [2.174]$$

These account for the fact that bending occurs with respect to an axis passing through the centroid G, while torsion occurs about the normal axis through the center of twist C, located by the vector $\overrightarrow{GC} := \bar{\mathbf{r}}_C = r_{2C}\bar{\mathbf{a}}_2 + r_{3C}\bar{\mathbf{a}}_3$, of modulus $r_C := \|\bar{\mathbf{r}}_C\|$. When the previous procedure restarts, we find the following constitutive law:

$$\begin{aligned} N &= \mathring{N} + EA\varepsilon + \frac{1}{2}EJ_C\kappa_1^2 \\ T_2 &= \mathring{T}_2 + GA_2\gamma_2 \\ T_3 &= \mathring{T}_3 + GA_3\gamma_3 \\ M_1 &= \mathring{M}_1 + GJ_1^\star\kappa_1 + EJ_C\varepsilon\kappa_1 - EY_{3C}\kappa_3\kappa_1 + EY_{2C}\kappa_2\kappa_1 \\ M_2 &= \mathring{M}_2 + EJ_2\kappa_2 + \frac{1}{2}EY_{2C}\kappa_1^2 \\ M_3 &= \mathring{M}_3 + EJ_3\kappa_3 - \frac{1}{2}EY_{3C}\kappa_1^2 \end{aligned} \qquad [2.175]$$

where all the inertias are evaluated with respect to C:

$$J_C := \int_{\mathcal{A}} \left(r - r_C\right)^2 dA, \qquad Y_{2C} := \int_{\mathcal{A}} r_3 \left(r - r_C\right)^2 dA$$
$$Y_{3C} := \int_{\mathcal{A}} r_2 \left(r - r_C\right)^2 dA \qquad [2.176]$$

and, moreover:

$$GJ_1^\star := GJ_1 + \mathring{N}\rho_C^2 + \mathring{M}_2\frac{Y_{2C}}{J_2} - \mathring{M}_3\frac{Y_{3C}}{J_3} \qquad [2.177]$$

with $\rho_C^2 := J_C/A$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2.3.3 Homogenization of beam-like structures

There exist structures that are not really beams, but that can be roughly assimilated to be beams. Examples include: trussed beams, Vierendel beams, helicoidal springs and, in general, *periodic structures* in which one dimension is prevalent on the other two, realized as an assembly of a large number of *cells* that repeat themselves and therefore also called *cellular structures*. If we look at these structures on a large scale, and we are not interested in evaluating the local behavior, affecting one or few cells, but the global behavior interesting a large number of cells, then we can model the beam-like structure as a beam. The key point consists of a *homogenization* process of the cellular structure able to provide a constitutive law for the *equivalent* 1D-beam.[47] We will explain the procedure by referring to a specific exampie.

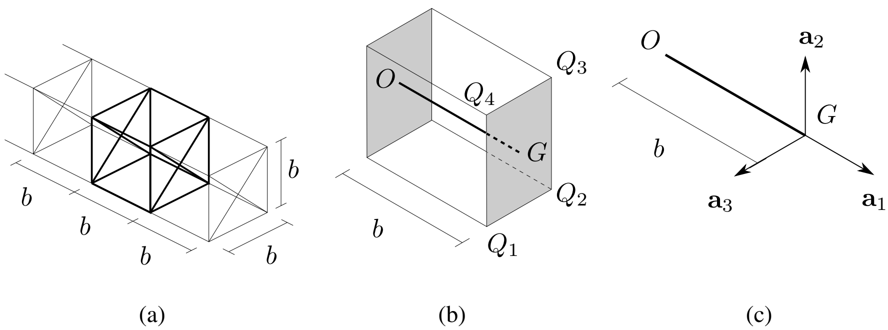


Figure 2.12: Trussed beam (refined model) and homogeneous beam (coarse model).

Let us consider the trussed beam of Figure 2.12, made up of several cubic cells of side b, each made up of pinned trusses of known axial stiffness. We will refer to the trussed beam as the *refined model* and to the homogeneous beam as the *coarse model*. We consider one of the cells as a *representative volume* of the periodic structure, and we take a piece of equal length to the homogeneous beam. Our goal is *equating the elastic potentials stored in the two volumes when the two models undergo the same displacements*. Of course, we cannot enforce the equality of the displacement field in each interior point, since the models are different, but we can equate the displacements of “sections” of the two models, by identifying the cross-stiffened meshes of the trussed beam as the cross-sections of the homogeneous beam. Moreover, since our coarse model is unable to account for dilatation of the section, we have to assume that the cross-stiffened sections are undeformable and behave as rigid bodies. Which displacements should be assigned to the sections? The best way is to assign a *uniform strain field* inside the volume of the coarse model and then to evaluate the displacements at the cross-section. The assumption seems acceptable in view of the fact that we are interested in the behavior of a beam much longer than b.

47. Homogenization, however, can lead either to first-gradient beams [DIC 90] or higher-gradient beams [ALI 03, SEP 11]; here we limit ourselves to the former case.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

With these ideas in mind, we establish the following homogenization procedure:

1) Take the strains constant over the length b of the coarse model, delimited by the sections O and G, assume O fixed and evaluate the displacement $\mathbf{u} := \mathbf{u}_G$ and rotation $\mathbf{R} := \mathbf{R}_G$ of the section G, *by integrating the strain–displacement relationships* (in scalar form, equations [2.52] and [2.53]) in which strains are taken constant over s.

2) Attribute the displacement $\mathbf{u}$ and the rotation $\mathbf{R}$ to the right sections of the refined representative volume, and, by using $\mathbf{u}_{Q_k} = \mathbf{u} + (\mathbf{R} - \mathbf{I})\,\bar{\mathbf{r}}_{Q_k}$ (with $\bar{\mathbf{r}}_{Q_k} := \overrightarrow{GQ_k}$), evaluate the displacement $\mathbf{u}_{Q_k}$ of any points Q_k on the section through G, where a truss is connected.

3) Evaluate the unit extensions e_k of the trusses, their elastic potential energy $U_k(e_k){:} = \phi_k(e_k) l_k$ (with l_k the length) and then sum the energies as $U{:} = \sum_k U_k(e_k)$.

4) Evaluate the potential energy per unit length of the coarse model as $\phi := U/b$ and finally derive the constitutive law via equation [2.151].

Of course, since each step involves nonlinear analysis, the constitutive law will also be nonlinear. This could be tackled by a perturbation analysis, mainly to integrate the kinematic problem, which, however, will not be addressed in this book.

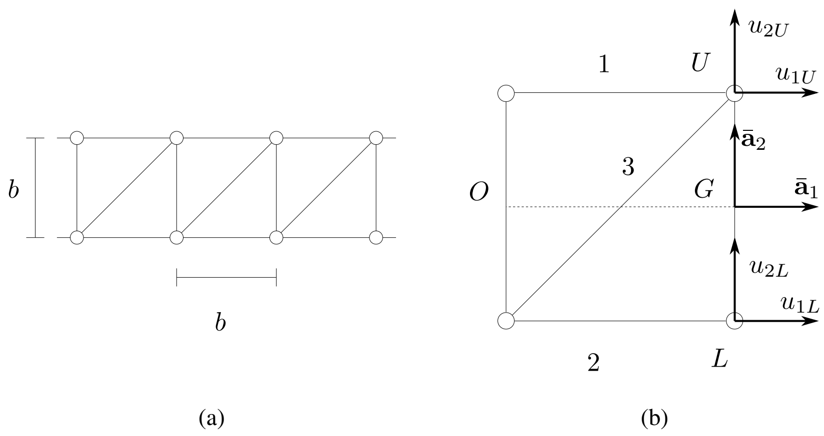


Figure 2.13: Planar truss.

As a simple example, let us consider the squared mesh planar truss in Figure 2.13 and carry out the analysis in the linear field. The strain–displacement relationships [2.52] and [2.53] reduce to:

$$\varepsilon = u_1', \qquad \gamma = u_2' - \theta, \qquad \kappa = \theta'$$

where $\theta := \theta_3, \gamma := \gamma_2, \kappa := \kappa_3$. Integration in the interval $s \in [0, b]$, with zero conditions on the left side of the cell ($s = 0$), provides the displacements on the right ($s = b$):

$$u_1 = b\varepsilon, \qquad u_2 = b\gamma + \frac{1}{2}b^2\kappa, \qquad \theta = b\kappa$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Here, the displacements of the upper and lower U, L nodes, whose distances from G are $\pm b/2$, are:

$$u_{1U,L} = b\varepsilon \mp \frac{1}{2}b^2\kappa, \qquad u_{2U,L} = b\gamma + \frac{1}{2}b^2\kappa$$

These displacements induce in the trusses the following strains:

$$e_{1,2} = \frac{u_{1U,L}}{b} = \varepsilon \mp \frac{1}{2}b\kappa, \qquad e_3 = \frac{\frac{\sqrt{2}}{2}\left(u_{1U} + u_{2U}\right)}{\sqrt{2}b} = \frac{1}{2}\left(\varepsilon + \gamma\right)$$

The elastic energy stored in the cell is $U = \frac{1}{2}EA_0\sum_{i=1}^{3} e_i^2 l_i$, where A_0 is the cross-sectional area of all trusses; by dividing it by the cell length, the elastic potential (per unit length) $\phi = U/b$ is derived:

$$\phi = \frac{1}{2}EA_0\left[\frac{5}{4}\varepsilon^2 + \frac{1}{4}\gamma^2 + \frac{\sqrt{2}}{2}\varepsilon\gamma + \frac{1}{2}b^2\kappa^2\right]$$

This is assumed as the elastic potential for the homogeneous beam. Finally, by applying the Green law [2.151], we find the stresses:

$$\begin{pmatrix} N \\ T \\ M \end{pmatrix} = EA_0\begin{bmatrix} \frac{5}{4} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{1}{4}b^2 & 0 \\ 0 & 0 & \frac{1}{2}b^2 \end{bmatrix}\begin{pmatrix} \varepsilon \\ \gamma \\ \kappa \end{pmatrix}$$

Note that longitudinal and transverse strains are coupled, due to the presence of the diagonal truss, which breaks the symmetry with respect to the beam axis.

2.3.4 Linear viscoelastic laws

Viscoelasticity accounts for slow phenomena, in which the response of the material is not instantaneous, as happens in elasticity, but evolves even when the solicitation remains constant in time. As peculiar examples, a *relaxation* occurs when the stress evolves, notwithstanding the strain is kept constant; conversely, a *creep* manifests itself when the strain evolves while the stress is constant. Such phenomena are intrinsically irreversible, so that they always entail a loss of energy[48].

Rheological models

The simplest rheological model accounting for linear viscosity is the Newton model (see Figure 2.14(a)), or dashpot, whose constitutive law expresses

48. Viscoelastic models are also important in dynamics if we want to model the internal dissipation of material. This leads to the concept of *internal damping* as opposited to that of "external damping", which only accounts for the interaction of the body with the surrounding air.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

proportionality between stress and strain-rate, i.e. $\sigma = \eta\dot{\varepsilon}$, where η is the *viscosity constant*; according to it, creep is unbounded since $\varepsilon \to \infty$ when $\sigma = \text{const}$. If, however, a linear spring is placed in parallel to the dashpot, the Kelvin–Voigt (KV) rheological model is obtained (Figure 2.14(b)), for which the stress is the sum of: (a) an elastic part, $\sigma_e = E\varepsilon$, where E is a modulus of *delayed elasticity* and (b) a viscous part, $\sigma_v = \eta\dot{\varepsilon}$; hence:

$$\sigma = E\varepsilon + \eta\dot{\varepsilon} \tag{2.178}$$

Such a model is able to describe, at least qualitatively, the creep phenomenon, predicting a finite asymptotic response $\varepsilon \to \sigma/E$, reached via an exponential evolution[49]. However, the KV model is unable to describe the instantaneous elastic response, and therefore a second elastic spring of modulus E_0 (called of *instantaneous elasticity*) is placed in series with it, to provide the *standard* (or three-parameter) model (Figure 2.14(c))[50]:

$$\dot{\sigma} + \frac{E_0 + E_v}{\eta}\sigma = E_0\dot{\varepsilon} + \frac{E_0 E_v}{\eta}\varepsilon \tag{2.179}$$

The standard model predicts the asymptotic response to creep as $\varepsilon \to \sigma(1/E_v + 1/E_0)$.

For its simplicity, the KV model is very often preferred to the standard model, mainly in dynamics. If the former is used, it is suitable to take $1/E = (1/E_v + 1/E_0)$, so that the asymptotic response of the standard model is correctly captured.

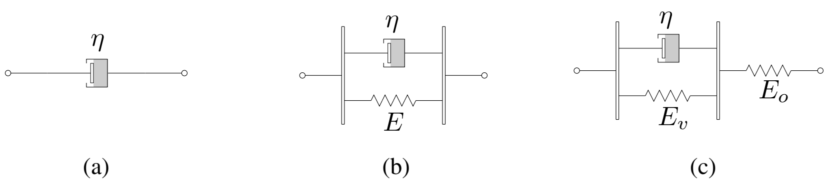


Figure 2.14: Viscous and viscoelastic rheological models: (a) Newton, (b) Kelvin–Voigt and (c) standard.

Viscoelastic beams

Rheological models suggest formulation of viscoelastic laws for continuous systems. If the KV model is used, consistently with equation [2.178], the relevant

49. To find steady solutions, here and equation [2.179], put all time-derivatives equal to zero here and in equation [2.179].

50. The standard model is governed by $\sigma = E_v\varepsilon_{KV} + \eta\dot{\varepsilon}_{KV}$, concerning the Kelvin–Voigt model, $\sigma = E_0\varepsilon_0$, relevant to the elastic spring, and, finally, the compatibility condition $\varepsilon = \varepsilon_{KV} + \varepsilon_0$, from which equation [2.179] is drawn.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

constitutive laws are obtained from the elastic laws by exploiting the similarity of the viscous terms. Thus, the linear uncoupled elastic law [2.164] modifies into[51]:

$$\begin{aligned} N &= \overset{\circ}{N} + EA\varepsilon + \eta A\dot{\varepsilon} \\ T_j &= \overset{\circ}{T}_j + GA_j\gamma_j + \zeta A_j\dot{\gamma}_j \\ M_1 &= \overset{\circ}{M}_1 + GJ_1\kappa_1 + \zeta J_1\dot{\kappa}_1 \\ M_j &= \overset{\circ}{M}_j + EJ_j\kappa_j + \eta J_j\dot{\kappa}_j \end{aligned} \qquad [2.180]$$

with $j = 2, 3$. Here, by accounting for viscosity, the elastic constants E, G were substituted by the viscous coefficients η, ζ and strains by strain-rates.

Under the same hypotheses, according to the standard model, we have:

$$\begin{aligned} \dot{N} + \frac{E_0 + E_v}{\eta}\left(N - \overset{\circ}{N}\right) &= E_0 A\dot{\varepsilon} + \frac{E_0 E_v}{\eta} A\varepsilon \\ \dot{T}_j + \frac{G_0 + G_v}{\zeta}\left(T_j - \overset{\circ}{T}_j\right) &= G_0 A_j\dot{\gamma}_j + \frac{G_0 G_v}{\zeta} A_j\gamma_j \\ \dot{M}_1 + \frac{G_0 + G_v}{\zeta}\left(M_1 - \overset{\circ}{M}_1\right) &= G_0 J_1\dot{\kappa}_1 + \frac{G_0 G_v}{\zeta} J_1\kappa_1 \\ \dot{M}_j + \frac{E_0 + E_v}{\eta}\left(M_i - \overset{\circ}{M}_i\right) &= E_0 J_j\dot{\kappa}_i + \frac{E_0 E_v}{\eta} J_j\kappa_i \end{aligned} \qquad [2.181]$$

2.4 The Fundamental Problem

The Fundamental Problem of continuum mechanics, when referring to the beam, is formulated as follows. "A beam is given, which is straight in its reference configuration, whose material is hyperelastic or viscoelastic. The beam is (fully or partially) constrained at the boundaries A, B, where displacements are prescribed, and loaded by known forces and couples, distributed along the centerline and/or on the portion of the boundary which is free of constraints. We want to evaluate the state of stress, of strain and the actual configuration of the beam (as a function of time in the dynamical problem, or independent of time in the static problem)".

The data of the problem are therefore (a) the displacements $\breve{\boldsymbol{u}}$, $\breve{\boldsymbol{\theta}}$ prescribed at the constraints and (b) the forces $\bar{\boldsymbol{p}}$, $\bar{\boldsymbol{c}}$ in the field and the forces at the boundaries $\bar{\boldsymbol{P}}_H$, $\bar{\boldsymbol{C}}_H$, $(H = A, B)$, all expressed in the reference basis. The unknowns are: (a) the displacements $\boldsymbol{u}$ and the rotations $\boldsymbol{\theta}$; (b) the strains $\boldsymbol{e}$ and the curvatures $\boldsymbol{k}$; and (c) the force-stresses $\boldsymbol{t}$ and the couple-stresses $\boldsymbol{m}$.

51. For the more general linear coupled law [2.157], the constitutive law becomes $\boldsymbol{\sigma} = \overset{\circ}{\boldsymbol{\sigma}} + \boldsymbol{E}\boldsymbol{\varepsilon} + \boldsymbol{H}\dot{\boldsymbol{\varepsilon}}$, where $\boldsymbol{E}$ is the elastic matrix and $\boldsymbol{H}$ is the *viscosity matrix.*

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2.4.1 Exact equations

The governing equations

The equations governing the problem, here rewritten in matrix form, are:

– six strain–displacement relationships [2.51], [2.54]:

$$\begin{aligned} \boldsymbol{e} &= \boldsymbol{R}^T \left(\bar{\boldsymbol{a}}_1 + \boldsymbol{u}'\right) - \bar{\boldsymbol{a}}_1 \\ \boldsymbol{k} &= \bar{\boldsymbol{B}}_\omega \boldsymbol{\theta}' \end{aligned} \qquad [2.182]$$

– six balance equations in the current configuration [2.136] and [2.138]:

$$\begin{aligned} &(\boldsymbol{t}' + \boldsymbol{K}\boldsymbol{t}) + \boldsymbol{R}^T \bar{\boldsymbol{p}} = m\boldsymbol{R}^T \ddot{\boldsymbol{u}} \\ &(\boldsymbol{m}' + \boldsymbol{K}\boldsymbol{m}) + \boldsymbol{\Lambda}\boldsymbol{t} + \boldsymbol{R}^T \bar{\boldsymbol{c}} = \boldsymbol{I}_G \dot{\boldsymbol{\omega}} \end{aligned} \qquad [2.183]$$

or, equivalently, in the reference configuration [2.141]:

$$\begin{aligned} &\boldsymbol{R}\left(\boldsymbol{t}' + \boldsymbol{K}\boldsymbol{t}\right) + \bar{\boldsymbol{p}} = m\ddot{\boldsymbol{u}} \\ &\boldsymbol{R}\left(\boldsymbol{m}' + \boldsymbol{K}\boldsymbol{m}\right) + \boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{t} + \bar{\boldsymbol{c}} = \boldsymbol{R}\boldsymbol{I}_G \dot{\boldsymbol{\omega}} \end{aligned} \qquad [2.184]$$

– six constitutive laws, e.g. elastic [2.156]:

$$\begin{aligned} \boldsymbol{t} &= \mathring{\boldsymbol{t}} + \boldsymbol{E}_{ee}\boldsymbol{e} + \boldsymbol{E}_{ek}\boldsymbol{k} \\ \boldsymbol{m} &= \mathring{\boldsymbol{m}} + \boldsymbol{E}_{ke}\boldsymbol{e} + \boldsymbol{E}_{kk}\boldsymbol{k} \end{aligned} \qquad [2.185]$$

– six geometric and/or mechanical boundary conditions, to be enforced at the ends of the beam (equations [2.19] and [2.142]):

$$\boldsymbol{u} = \breve{\boldsymbol{u}}, \qquad \boldsymbol{\theta} = \breve{\boldsymbol{\theta}} \qquad [2.186]$$

and, in the current configuration:

$$\mp\, \boldsymbol{t}_H = \boldsymbol{R}_H^T \bar{\boldsymbol{P}}_H, \qquad \mp \boldsymbol{m}_H = \boldsymbol{R}_H^T \bar{\boldsymbol{C}}_H \qquad [2.187]$$

or, in the reference configuration:

$$\mp\, \boldsymbol{R}_H \boldsymbol{t}_H = \bar{\boldsymbol{P}}_H, \qquad \mp \boldsymbol{R}_H \boldsymbol{m}_H = \bar{\boldsymbol{C}}_H \qquad [2.188]$$

Overall, we have 18 field scalar equations containing 18 unknowns.

The equations of motion for unprestressed beams

The governing equations are combined to express the balance equations in terms of displacements. Equations in the current configuration are found to be much shorter than those in the reference configuration, and therefore the former are preferred here. When a diagonal elastic law is adopted ($\boldsymbol{E}_{ek} = \boldsymbol{E}_{ke} = \boldsymbol{0}$), and prestress is absent they read:

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

– balance of linear momentum:

$$
\begin{aligned}
&\{EA[(1+u_1')\cos\theta_2\cos\theta_3+u_2'\cos\theta_2\sin\theta_3-u_3'\sin\theta_2-1]\}'\\
&\quad-GA_2[(1+u_1')(\sin\theta_1\sin\theta_2\cos\theta_3-\cos\theta_1\sin\theta_3)\\
&\quad+u_2'(\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3)\\
&\quad+u_3'(\sin\theta_1\cos\theta_2)][\theta_3'\cos\theta_1\cos\theta_2-\theta_2'\sin\theta_1]\\
&\quad+GA_3[(1+u_1')(\cos\theta_1\sin\theta_2\cos\theta_3\\
&\quad+\sin\theta_1\sin\theta_3)+u_2'(\cos\theta_1\sin\theta_2\sin\theta_3\\
&\quad-\cos\theta_3\sin\theta_1)+u_3'(\cos\theta_1\cos\theta_2)][\theta_2'\cos\theta_1+\theta_3'\sin\theta_1\cos\theta_2]\\
&\quad-(m\ddot{u}_1-\bar{p}_1)[\cos\theta_1\cos\theta_3]-(m\ddot{u}_2-\bar{p}_2)\cos\theta_2\sin\theta_3\\
&\quad+(m\ddot{u}_3-\bar{p}_3)\sin\theta_2=0\\
&\{GA_2[(1+u_1')(\sin\theta_1\sin\theta_2\cos\theta_3-\cos\theta_1\sin\theta_3)\\
&\quad+u_2'(\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3)+u_3'(\sin\theta_1\cos\theta_2)]\}'\\
&\quad+EA[(1+u_1')\cos\theta_2\cos\theta_3+u_2'\cos\theta_2\sin\theta_3-u_3'\sin\theta_2\\
&\quad-1][\theta_3'\cos\theta_1\cos\theta_2-\theta_2'\sin\theta_1]\\
&\quad-GA_3[(1+u_1')(\cos\theta_1\sin\theta_2\cos\theta_3\\
&\quad+\sin\theta_1\sin\theta_3)+u_2'(\cos\theta_1\sin\theta_2\sin\theta_3-\sin\theta_1\cos\theta_3)\\
&\quad+u_3'(\cos\theta_1\cos\theta_2)][\theta_1'-\theta_3\sin\theta_2']\\
&\quad-(m\ddot{u}_1-\bar{p}_1)[\sin\theta_1\sin\theta_2\cos\theta_3-\cos\theta_1\sin\theta_3]\\
&\quad-(m\ddot{u}_2-\bar{p}_2)[\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3]\\
&\quad-(m\ddot{u}_3-\bar{p}_3)[\sin\theta_1\cos\theta_2]=0\\
&\{GA_3[(1+u_1')(\cos\theta_1\sin\theta_2\cos\theta_3+\sin\theta_1\sin\theta_3)\\
&\quad+u_2'(\cos\theta_1\sin\theta_2\sin\theta_3-\sin\theta_1\cos\theta_3)\\
&\quad+u_3'(\cos\theta_1\cos\theta_2)]\}'-EA[(1+u_1')\cos\theta_2\cos\theta_3+u_2'\cos\theta_2\sin\theta_3\\
&\quad-u_3'\sin\theta_2-1][\theta_2'\cos\theta_1+\theta_3'\sin\theta_1\cos\theta_2]\\
&\quad+GA_2[(1+u_1')(\sin\theta_1\sin\theta_2\cos\theta_3-\cos\theta_1\sin\theta_3)\\
&\quad+u_2'(\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3)\\
&\quad+u_3'(\sin\theta_1\cos\theta_2)][\theta_1'-\theta_3'\sin\theta_2]\\
&\quad-(m\ddot{u}_1-\bar{p}_1)[\cos\theta_1\sin\theta_2\cos\theta_3+\sin\theta_1\sin\theta_3]\\
&\quad-(m\ddot{u}_2-\bar{p}_2)[\cos\theta_1\sin\theta_2\sin\theta_3-\sin\theta_1\cos\theta_3]\\
&\quad-(m\ddot{u}_3-\bar{p}_3)\cos\theta_1\cos\theta_2=0
\end{aligned}
\qquad [2.189]
$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

– balance of angular momentum:

$$
\begin{aligned}
&\{GJ_1[\theta_1' - \theta_3' \sin\theta_2]\}' + (GJ_3 - GJ_2)[\theta_2' \cos\theta_1 \\
&\quad + \theta_3' \sin\theta_1 \cos\theta_2][\theta_3' \cos\theta_1 \cos\theta_2 - \theta_2' \sin\theta_1] \\
&\quad + (GA_3 - GA_2)[(1 + u_1')(\sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3) \\
&\quad + u_2'(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3) \\
&\quad + u_3'(\sin\theta_1 \cos\theta_2)][(1 + u_1')(\cos\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad + \sin\theta_1 \sin\theta_3) + u_2'(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3) \\
&\quad + u_3'(\cos\theta_1 \cos\theta_2)] + \cos\theta_1 \cos\theta_3 \bar{c}_1 + \cos\theta_2 \sin\theta_3 \bar{c}_2 \\
&\quad - \sin\theta_2 \bar{c}_3 - I_1\{\dot\theta_1 - \theta_3' \sin\theta_2\}^{\cdot} = 0 \\
&\{GJ_2[\theta_2' \cos\theta_1 + \theta_3' \sin\theta_1 \cos\theta_2]\}' + (GJ_1 - GJ_3)[\theta_1' \\
&\quad - \theta_3' \sin\theta_2][\theta_3' \cos\theta_1 \cos\theta_2 - \theta_2' \sin\theta_1] \\
&\quad + (EA - GA_3)[(1 + u_1') \cos\theta_2 \cos\theta_3 + u_2' \cos\theta_2 \sin\theta_3 \\
&\quad - u_3' \sin\theta_2 - 1][(1 + u_1')(\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3) \\
&\quad + u_2'(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3) \\
&\quad + u_3'(\cos\theta_1 \cos\theta_2)] - GA_3[(1 + u_1')(\cos\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad + \sin\theta_1 \sin\theta_3) + u_2'(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3) \\
&\quad + u_3'(\cos\theta_1 \cos\theta_2)] + [\sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3]\bar{c}_1 \\
&\quad + [\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3]\bar{c}_2 + \sin\theta_1 \cos\theta_2 \bar{c}_3 \\
&\quad - I_2\{\dot\theta_2 \cos\theta_1 + \dot\theta_3 \sin\theta_1 \cos\theta_2\}^{\cdot} = 0 \\
&\{GJ_3[\theta_3' \cos\theta_1 \cos\theta_2 - \theta_2' \sin\theta_1]\}' + (GJ_2 - GJ_1)[\theta_1' \\
&\quad - \theta_3' \sin\theta_2][\theta_2' \cos\theta_1 + \theta_3' \sin\theta_1 \cos\theta_2] \\
&\quad + (GA_2 - EA)[(1 + u_1') \cos\theta_2 \cos\theta_3 + u_2' \cos\theta_2 \sin\theta_3 \\
&\quad - u_3' \sin\theta_2 - 1][(1 + u_1')(\sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad - \cos\theta_1 \sin\theta_3) + u_2'(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3) \\
&\quad + u_3'(\sin\theta_1 \cos\theta_2)] + GA_2[(1 + u_1')(\sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3) \\
&\quad + u_2'(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3) \\
&\quad + u_3'(\sin\theta_1 \cos\theta_2)] + [\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3]\bar{c}_1 \\
&\quad + [\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3]\bar{c}_2 + \cos\theta_1 \cos\theta_2 \bar{c}_3 \\
&\quad - I_3\{\dot\theta_3 \cos\theta_1 \cos\theta_2 - \dot\theta_2 \sin\theta_1\}^{\cdot} = 0
\end{aligned}
\qquad [2.190]
$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

together with the mechanical boundary conditions:

$$
\begin{aligned}
&\mp EA[(1+u_1')\cos\theta_2\cos\theta_3 + u_2'\cos\theta_2\sin\theta_3 - u_3'\sin\theta_2 - 1]_H \\
&\quad = \cos\theta_1\cos\theta_3\bar{P}_{1H} + \cos\theta_2\sin\theta_3\bar{P}_{2H} - \sin\theta_2\bar{P}_{3H} \\
&\mp GA_2[(1+u_1')(\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3) \\
&\quad + u_2'(\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3) + u_3'(\sin\theta_1\cos\theta_2)]_H \\
&\quad = [\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3]\bar{P}_{1H} \\
&\quad + [\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3]\bar{P}_{2H} + \sin\theta_1\cos\theta_2\bar{P}_{3H} \\
&\mp GA_3[(1+u_1')(\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3) \\
&\quad + u_2'(\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3) + u_3'(\cos\theta_1\cos\theta_2)]_H \\
&\quad = [\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3]\bar{P}_{1H} \\
&\quad + [\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3]\bar{P}_{2H} + \cos\theta_1\cos\theta_2\bar{P}_{3H} \\
&\mp GJ_1[\theta_1' - \theta_3'\sin\theta_2]_H \\
&\quad = \cos\theta_1\cos\theta_3\bar{C}_{1H} + \cos\theta_2\sin\theta_3\bar{C}_{2H} - \sin\theta_2\bar{C}_{3H} \\
&\mp GJ_2[\theta_2'\cos\theta_1 + \theta_3'\sin\theta_1\cos\theta_2]_H \\
&\quad = [\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3]\bar{C}_{1H} + [\cos\theta_1\cos\theta_3 \\
&\quad + \sin\theta_1\sin\theta_2\sin\theta_3]\bar{C}_{2H} + \sin\theta_1\cos\theta_2\bar{C}_{3H} \\
&\mp GJ_3[\theta_3'\cos\theta_1\cos\theta_2 - \theta_2'\sin\theta_1]_H \\
&\quad = [\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3]\bar{C}_{1H} \\
&\quad + [\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3]\bar{C}_{2H} + \cos\theta_1\cos\theta_2\bar{C}_{3H}
\end{aligned}
\qquad [2.191]
$$

and/or the geometric boundary conditions [2.186].

Since the equations are nonlinear, uniqueness of the solution is not ensured. The problem, in general, cannot be solved in closed form, but it calls for numerical algorithms. A different approach consists of tackling the equation by perturbation methods, by looking for solutions that are perturbations of the (unique) solution of the associated linear problem.

2.4.2 The linearized theory for elastic prestressed beams

As discussed with the metamodel (section 1.4), sometimes a linear model, accounting for prestress, is suited to the scope (e.g., to determine a bifurcation load). In these cases, the *linearized theory* is employed, in which the equations of motion are linearized in the displacements, but the prestresses are assumed as finite

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

quantities (while the increment of stresses are considered as first-order quantities). Thus, for example, the axial force $N = \mathring{N} + EA\varepsilon$ is made up of a (order-1) leading part $\mathring{N}$ and an (first-order) infinitesimal correction $\tilde{N} := EA\varepsilon$. Accordingly, kinematics is linearized (since higher power displacements must be neglected), but the balance equations must account for terms which are *bilinear in strains and prestresses*. As an example, a term such as $\kappa_3 N$, appearing in the equilibrium equations, since $\kappa_3 = \theta_3' + \text{h.o.t.}$, must be approximated as $\mathring{N}\theta_3'$ in the linearized theory. Since the linearized equations are much more manageable than the nonlinear equations, they are convenientely expressed in the reference basis.

According to these ideas, kinematics states that:

$$\begin{aligned} \varepsilon &= u_1' \\ \gamma_2 &= u_2' - \theta_3 \\ \gamma_3 &= u_3' + \theta_2 \\ \kappa_1 &= \theta_1' \\ \kappa_2 &= \theta_2' \\ \kappa_3 &= \theta_3' \end{aligned} \qquad [2.192]$$

However, the balance in the adjacent configuration requires (linear momentum equations):

$$\begin{aligned} &(\mathring{N} + \tilde{N})' - \theta_3'\mathring{T}_2 + \theta_2'\mathring{T}_3 + \mathring{p}_1 + \tilde{p}_1 - m\ddot{u}_1 = 0 \\ &(\mathring{T}_2 + \tilde{T}_2)' - \theta_1'\mathring{T}_3 + \theta_3'\mathring{N} + \mathring{p}_2 + \tilde{p}_2 - m\ddot{u}_2 = 0 \\ &(\mathring{T}_3 + \tilde{T}_3)' + \theta_1'\mathring{T}_2 - \theta_2'\mathring{N} + \mathring{p}_3 + \tilde{p}_3 - m\ddot{u}_3 = 0 \end{aligned} \qquad [2.193]$$

and (angular momentum equations):

$$\begin{aligned} &(\mathring{M}_1 + \tilde{M}_1)' - \theta_3'\mathring{M}_2 + \theta_2'\mathring{M}_3 \\ &\quad - (u_3' + \theta_2)\mathring{T}_2 + (u_2' - \theta_3)\mathring{T}_3 + \mathring{c}_1 + \tilde{c}_1 - I_1\ddot{\theta}_1 = 0 \\ &(\mathring{M}_2 + \tilde{M}_2)' + \theta_3'\mathring{M}_1 - \theta_1\mathring{M}_3 \\ &\quad + (u_3' + \theta_2)\mathring{N} - u_1'\mathring{T}_3 - GA_2(u_2' - \theta_3) + \mathring{c}_2 + \tilde{c}_2 - I_2\ddot{\theta}_2 = 0 \\ &(\mathring{M}_3 + \tilde{M}_3)' + \theta_1'\mathring{M}_2 - \theta_2\mathring{M}_1 \\ &\quad + u_1'\mathring{T}_2 - (u_2' - \theta_3)\mathring{N} + GA_3(u_3' + \theta_2) + \mathring{c}_3 + \tilde{c}_3 - I_3\ddot{\theta}_3 = 0 \end{aligned} \qquad [2.194]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with the boundary conditions:

$$\begin{aligned}
&\mp[\overset{\circ}{N}+\tilde{N}-\theta_3(\overset{\circ}{T}_2+\tilde{T}_2)+\theta_2(\overset{\circ}{T}_3+\tilde{T}_3)]_H=\overset{\circ}{P}_{1H}+\tilde{P}_{1H}\\
&\mp[\overset{\circ}{T}_2+\tilde{T}_2+\theta_3(\overset{\circ}{N}+\tilde{N})-\theta_1(\overset{\circ}{T}_3+\tilde{T}_3)]_H=\overset{\circ}{P}_{2H}+\tilde{P}_{2H}\\
&\mp[\overset{\circ}{T}_3+\tilde{T}_3-\theta_2(\overset{\circ}{N}+\tilde{N})+\theta_1(\overset{\circ}{T}_2+\tilde{T}_2)]_H=\overset{\circ}{P}_{3H}+\tilde{P}_{3H}\\
&\mp[\overset{\circ}{M}_1+\tilde{M}_1-\theta_3(\overset{\circ}{M}_2+\tilde{M}_2)+\theta_2(\overset{\circ}{M}_3+\tilde{M}_3)]_H=\overset{\circ}{C}_{1H}+\tilde{C}_{1H}\\
&\mp[\overset{\circ}{M}_2+\tilde{M}_2+\theta_3(\overset{\circ}{M}_1+\tilde{M}_1)-\theta_1(\overset{\circ}{M}_3+\tilde{M}_3)]_H=\overset{\circ}{C}_{2H}+\tilde{C}_{2H}\\
&\mp[\overset{\circ}{M}_3+\tilde{M}_3-\theta_2(\overset{\circ}{M}_1+\tilde{M}_1)+\theta_1(\overset{\circ}{M}_2+\tilde{M}_2)]_H=\overset{\circ}{C}_{3H}+\tilde{C}_{3H}
\end{aligned} \quad [2.195]$$

Here, both stresses and loads (assumed to be gravitational) have been expressed in their incremental form, e.g. $N=\overset{\circ}{N}+\tilde{N}$, $\mathbf{p}=\overset{\circ}{\mathbf{p}}+\tilde{\mathbf{p}}$, where the symbol $(\overset{\circ}{\ })$ denotes prestresses and preloads and the symbol $(\tilde{\ })$ denotes a small increment. However, taking into account that prestresses and preloads are *equilibrated* in the reference configuration (in which displacements and strains disappear), it follows that terms exclusively containing over-ring quantities can be mutually cancelled. Therefore, the former equilibrium equations reduce to:

$$\begin{aligned}
&\tilde{N}'-\theta_3'\overset{\circ}{T}_2+\theta_2'\overset{\circ}{T}_3+\tilde{p}_1-m\ddot{u}_1=0\\
&\tilde{T}_2'-\theta_1'\overset{\circ}{T}_3+\theta_3'\overset{\circ}{N}+\tilde{p}_2-m\ddot{u}_2=0\\
&\tilde{T}_3'+\theta_1'\overset{\circ}{T}_2-\theta_2'\overset{\circ}{N}+\tilde{p}_3-m\ddot{u}_3=0
\end{aligned} \quad [2.196]$$

and:

$$\begin{aligned}
&\tilde{M}_1'-\theta_3'\overset{\circ}{M}_2+\theta_2'\overset{\circ}{M}_3-(u_3'+\theta_2)\overset{\circ}{T}_2+(u_2'-\theta_3)\overset{\circ}{T}_3+\tilde{c}_1-I_1\ddot{\theta}_1=0\\
&\tilde{M}_2'-GA_2(u_2'-\theta_3)+\theta_3'\overset{\circ}{M}_1-\theta_1\overset{\circ}{M}_3+(u_3'+\theta_2)\overset{\circ}{N}-u_1'\overset{\circ}{T}_3+\tilde{c}_2-I_2\ddot{\theta}_2=0\\
&\tilde{M}_3'+GA_3(u_3'+\theta_2)+\theta_1'\overset{\circ}{M}_2-\theta_2\overset{\circ}{M}_1+u_1'\overset{\circ}{T}_2-(u_2'-\theta_3)\overset{\circ}{N}+\tilde{c}_3-I_3\ddot{\theta}_3=0
\end{aligned} \quad [2.197]$$

while boundary conditions become:

$$\begin{aligned}
&\mp[\tilde{N}-\theta_3\overset{\circ}{T}_2+\theta_2\overset{\circ}{T}_3]_H=\tilde{P}_{1H}\\
&\mp[\tilde{T}_2+\theta_3\overset{\circ}{N}-\theta_1\overset{\circ}{T}_3]_H=\tilde{P}_{2H}\\
&\mp[\tilde{T}_3-\theta_2\overset{\circ}{N}+\theta_1\overset{\circ}{T}_2]_H=\tilde{P}_{3H}\\
&\mp[\tilde{M}_1-\theta_3\overset{\circ}{M}_2+\theta_2\overset{\circ}{M}_3]_H=\tilde{C}_{1H}\\
&\mp[\tilde{M}_2+\theta_3\overset{\circ}{M}_1-\theta_1\overset{\circ}{M}_3]_H=\tilde{C}_{2H}\\
&\mp[\tilde{M}_3-\theta_2\overset{\circ}{M}_1+\theta_1\overset{\circ}{M}_2]_H=\tilde{C}_{3H}
\end{aligned} \quad [2.198]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Equations [2.196] and [2.197] are known as *incremental equilibrium equations*.

When the linear elastic law is used for the incremental stresses, the following equations of motion are found:

$$
\begin{aligned}
&(EAu_1')' - \theta_3'\mathring{T}_2 + \theta_2'\mathring{T}_3 + \tilde{p}_1 - m\ddot{u}_1 = 0\\
&(GA_2(u_2' - \theta_3))' - \theta_1'\mathring{T}_3 + \theta_3'\mathring{N} + \tilde{p}_2 - m\ddot{u}_2 = 0\\
&(GA_3(u_3' - \theta_2))' + \theta_1'\mathring{T}_2 - \theta_2'\mathring{N} + \tilde{p}_3 - m\ddot{u}_3 = 0
\end{aligned}
\qquad [2.199]
$$

and:

$$
\begin{aligned}
&(GJ_1\theta_1)' - \theta_3'\mathring{M}_2 + \theta_2'\mathring{M}_3 - (u_3' + \theta_2)\mathring{T}_2 + (u_2' - \theta_3)\mathring{T}_3 + \tilde{c}_1 - I_1\ddot{\theta}_1 = 0\\
&(GJ_2\theta_2)' - GA_2(u_2' - \theta_3) + \theta_3'\mathring{M}_1 - \theta_1\mathring{M}_3\\
&\quad + (u_3' + \theta_2)\mathring{N} - u_1'\mathring{T}_3 + \tilde{c}_2 - I_2\ddot{\theta}_2 = 0\\
&(GJ_3\theta_3)' + GA_3(u_3' + \theta_2) + \theta_1'\mathring{M}_2 - \theta_2\mathring{M}_1 + u_1'\mathring{T}_2\\
&\quad - (u_2' - \theta_3)\mathring{N} + \tilde{c}_3 - I_3\ddot{\theta}_3 = 0
\end{aligned}
\qquad [2.200]
$$

with the boundary conditions:

$$
\begin{aligned}
&\mp [EAu_1' - \theta_3\mathring{T}_2 + \theta_2\mathring{T}_3]_H = \tilde{P}_{1H}\\
&\mp [GA_2(u_2' - \theta_3) + \theta_3\mathring{N} - \theta_1\mathring{T}_3]_H = \tilde{P}_{2H}\\
&\mp [GA_3(u_3' - \theta_2) - \theta_2\mathring{N} + \theta_1\mathring{T}_2]_H = \tilde{P}_{3H}\\
&\mp [GJ_1\theta_1 - \theta_3\mathring{M}_2 + \theta_2\mathring{M}_3]_H = \tilde{C}_{1H}\\
&\mp [GJ_2\theta_2 + \theta_3\mathring{M}_1 - \theta_1\mathring{M}_3]_H = \tilde{C}_{2H}\\
&\mp [GJ_3\theta_3 - \theta_2\mathring{M}_1 + \theta_1\mathring{M}_2]_H = \tilde{C}_{3H}
\end{aligned}
\qquad [2.201]
$$

They are the equations governing the (small) motion around a prestressed equilibrium configuration[52].

REMARK 2.20. While the linear theory expresses the equilibrium in the reference configuration, the linearized theory enforces equilibrium in the so-called *adjacent configuration*, i.e. in a current configuration infinitely close to the former. Bilinear terms in strains and prestresses (of the type $\mathring{N}\theta_3'$) account for the effect of the (infinitesimal) change of configuration, leading to the appearance of a *geometric stiffness* of the beam.

52. These equations are of type $\boldsymbol{Lw} + \boldsymbol{Gw} = \tilde{\boldsymbol{p}}$ in the field and $\boldsymbol{\mathcal{L}}_H\boldsymbol{w} + \boldsymbol{\mathcal{G}}_H\boldsymbol{w} = \tilde{\boldsymbol{P}}$ on the boundary which we have already discussed in section 1.4.2 (equation [1.72]). Here, $\boldsymbol{L}, \boldsymbol{\mathcal{L}}_H$ are elastic stiffness operators and $\boldsymbol{G}, \boldsymbol{\mathcal{G}}_H$ are geometric stiffness operators.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

REMARK 2.21. When the prestress is dropped, the previous equations of motion reduce to the well-known equations of the *linear Timoshenko beam*.

2.5 The planar beam

Very often, straight beams are loaded in one of their principal inertia planes and are suitably constrained in such a way that they only stretch, shear and bend themselves in the same plane. In these cases, the beam is called *planar*, and the relevant problem is much more easily tackled. A *planar model* can, of course, be derived from the more general model in the 3D-space, addressed in the previous sections. Here, we will review the main steps of formulation, by specializing results to the planar case.

2.5.1 Kinematics

Let us consider a planar beam, of ends A, B, belonging to the plane π spanned by the $(\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2)$ unit vectors, and let $\bar{\mathcal{B}} = (\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3)$ be the orthogonal triad attached to the cross-sections in the reference configuration, with $\bar{\mathbf{a}}_1$ tangent to the centerline. Such a basis transforms into $\mathcal{B} = (\mathbf{a}_1(s,t), \mathbf{a}_2(s,t), \mathbf{a}_3)$ in the current configuration, with $\mathbf{a}_3 \equiv \bar{\mathbf{a}}_3$ independent of s, t.

Displacement and rotation

The displacement field is described by:

$$\mathbf{u} := u_1(s,t)\bar{\mathbf{a}}_1 + u_2(s,t)\bar{\mathbf{a}}_2 \quad [2.202]$$

and the rotation field by the tensor $\mathbf{R}(s,t)$, whose scalar representation in $\bar{\mathcal{B}}$ becomes:

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [2.203]$$

where $\theta := \theta_3$ denotes the amplitude of the rotation around $\bar{\mathbf{a}}_3$. Therefore, any geometric transformations depend on three scalar fields, $u_1(s,t), u_2(s,t), \theta(s,t)$, which constitute the configuration variables of the planar beam.

The previous equations must be sided by geometric boundary conditions of the type:

$$u_{1H} = \breve{u}_{1H}, \qquad u_{2H} = \breve{u}_{2H}, \qquad \theta_H = \breve{\theta}_H, \qquad H = A, B \quad [2.204]$$

where a curved overbar denotes a known quantity.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Strain–displacement relationships

Since $\mathbf{R}^T\mathbf{x}'$ belongs to π, the reference strain vector $\mathbf{e} \equiv \mathbf{e}_r := \mathbf{R}^T\mathbf{x}' - \bar{\mathbf{a}}_1$ also belongs to π. The current curvature vector $\mathbf{k}_c$, according to its definition $\mathbf{a}_3' = \mathbf{k}_c \times \mathbf{a}_3$, is orthogonal to π; therefore, the reference curvature vector $\mathbf{k} \equiv \mathbf{k}_r = \mathbf{R}^T\mathbf{k}_c$ also belongs to the plane. Accordingly:

$$\begin{aligned} \mathbf{e} &= \varepsilon\bar{\mathbf{a}}_1 + \gamma\bar{\mathbf{a}}_2 \\ \mathbf{k} &= \kappa\bar{\mathbf{a}}_3 \end{aligned} \qquad [2.205]$$

where $\gamma := \gamma_2$, $\kappa := \kappa_3$. Their scalar components are easily found to be [53]:

$$\begin{aligned} \varepsilon &= -1 + (1 + u_1')\cos\theta + u_2'\sin\theta \\ \gamma &= -(1 + u_1')\sin\theta + u_2'\cos\theta \\ \kappa &= \theta' \end{aligned} \qquad [2.206]$$

Velocity and spin

The velocity vector and the spin vector follow from equation [2.202] and from the analogy with the current curvature vector; they are given by:

$$\begin{aligned} \mathbf{v} &= \dot{u}_1(s,t)\bar{\mathbf{a}}_1 + \dot{u}_2(s,t)\bar{\mathbf{a}}_2 \\ \boldsymbol{\omega} &= \omega\mathbf{a}_3 = \dot{\theta}\mathbf{a}_3 \end{aligned} \qquad [2.207]$$

with $\omega \equiv \bar{\omega}$.

Strain rates

The time-derivatives of the strain components are the strain-rates:

$$\begin{aligned} \dot{\mathbf{e}} &= \dot{\varepsilon}\bar{\mathbf{a}}_1 + \dot{\gamma}_2\bar{\mathbf{a}}_2 \\ \dot{\mathbf{k}} &= \dot{\kappa}\bar{\mathbf{a}}_3 \end{aligned} \qquad [2.208]$$

---

53. As a matter of fact, since $\bar{\mathbf{a}}_1 + \mathbf{e} = \mathbf{R}^T(\bar{\mathbf{a}}_1 + \mathbf{u}')$, then:

$$\begin{pmatrix} 1+\varepsilon \\ \gamma \\ 0 \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1+u_1' \\ u_2' \\ 0 \end{pmatrix}$$

Moreover, since $\mathbf{K} = \mathbf{R}^T\mathbf{R}'$, then:

$$\boldsymbol{K} = \boldsymbol{R}^T\boldsymbol{R}' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\theta'\sin\theta & -\theta'\cos\theta & 0 \\ \theta'\cos\theta & -\theta'\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\theta' & 0 \\ \theta' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

They are related to the velocities through:

$$\dot{\varepsilon} = \dot{u}_1' \cos\theta + \dot{u}_2' \sin\theta + \gamma\dot{\theta}$$
$$\dot{\gamma} = -\dot{u}_1' \sin\theta + \dot{u}_2' \cos\theta - (1+\varepsilon)\dot{\theta} \qquad [2.209]$$
$$\dot{\kappa} = \dot{\theta}'$$

obtained by time-differentiating equations [2.206] and again using the same equations[54].

2.5.2 Dynamics

Let the beam be loaded by in-plane forces $\mathbf{p} = \bar{p}_1\bar{\mathbf{a}}_1 + \bar{p}_2\bar{\mathbf{a}}_2$ and in-plane couples $\mathbf{c} = \bar{c}\bar{\mathbf{a}}$, distributed along the beam, as well as forces and couples at boundaries $\mathbf{P}_H = \bar{P}_{1H}\bar{\mathbf{a}}_1 + \bar{P}_{2H}\bar{\mathbf{a}}_2$ and $\mathbf{C}_H = \bar{C}_H\bar{\mathbf{a}}_3$ with $H = A, B$.

Balance equations

The balance equations, in vector form, are still given by equations [2.107]:

$$\begin{aligned} &\mathbf{t}' + \mathbf{p} = m\ddot{\mathbf{u}} \\ &\mathbf{m}' + \mathbf{x}' \times \mathbf{t} + \mathbf{c} = \mathbf{I}_G\dot{\boldsymbol{\omega}} \end{aligned} \qquad [2.210]$$

and boundary conditions by equations [2.104]:

$$\begin{aligned} &(\mathbf{P}_H \pm \mathbf{t}_H)\cdot\mathbf{v}_H = 0 \\ &(\mathbf{C}_H \pm \mathbf{m}_H)\cdot\boldsymbol{\omega}_H, \qquad H = A, B \end{aligned} \qquad [2.211]$$

54. Equations [2.209] can be recast in matrix form as we did for the metamodel, namely $\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}(\boldsymbol{w}, \boldsymbol{w}')\,\dot{\boldsymbol{w}}$, where $\boldsymbol{\varepsilon} := (\varepsilon, \gamma, \kappa)^T$, $\boldsymbol{w} := (u_1, u_2, \theta)^T$ and:

$$\boldsymbol{D}\left(\boldsymbol{w}, \boldsymbol{w}'\right) := \begin{pmatrix} (\cos\theta)\,\partial_s & (\sin\theta)\,\partial_s & \gamma \\ -(\sin\theta)\,\partial_s & (\cos\theta)\,\partial_s & -(1+\varepsilon) \\ 0 & 0 & \partial_s \end{pmatrix}$$

in which both ε and γ depend on $\boldsymbol{w}$ and $\boldsymbol{w}'$. The matrix $\boldsymbol{D}(\boldsymbol{w}, \boldsymbol{w}')$ is the *kinematic operator* of the planar beam. If it is evaluated at the reference configuration, then $\boldsymbol{D}_0 := \boldsymbol{D}(\mathbf{0}, \mathbf{0})$ is the *infinitesimal kinematic operator* of the linear theory:

$$\boldsymbol{D}_0 := \begin{pmatrix} \partial_s & 0 & 0 \\ 0 & \partial_s & -1 \\ 0 & 0 & \partial_s \end{pmatrix}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Now, however, the force-stress $\mathbf{t}$ and the couple-stress $\mathbf{m}$ are an in-plane and an out-of-plane vector, respectively:

$$\begin{aligned} \mathbf{t} &= N\mathbf{a}_1 + T\mathbf{a}_2 \\ \mathbf{m} &= M\mathbf{a}_3 \end{aligned} \qquad [2.212]$$

where $T := T_2$ is the shear force and $M := M_3$ the bending moment; moreover, $\mathbf{I}_G\dot{\boldsymbol{\omega}} = I\ddot{\theta}\mathbf{a}_3$, with $I := I_3$ is the mass moment of inertia with respect to $\mathbf{a}_3$. When these equations are projected onto the reference basis, they become[55, 56]:

$$\begin{aligned} &(N' - \kappa T)\cos\theta - (T' + \kappa N)\sin\theta + \bar{p}_1 - m\ddot{u}_1 = 0 \\ &(N' - \kappa T)\sin\theta + (T' + \kappa N)\cos\theta + \bar{p}_2 - m\ddot{u}_2 = 0 \\ &M' + (1+\varepsilon)\,T - \gamma N + \bar{c} - I\ddot{\theta} = 0 \end{aligned} \qquad [2.213]$$

The balance equations must be sided by mechanical boundary conditions, of the type:

$$\begin{aligned} &\mp [N\cos\theta - T\sin\theta]_H = P_{1H} \\ &\mp [N\sin\theta + T\cos\theta]_H = \bar{P}_{2H} \\ &\mp M_H = \bar{C}_H, \qquad\qquad H = A, B \end{aligned} \qquad [2.214]$$

---

55. Indeed, by remembering the Poisson formula, and accounting for $\mathbf{k} = \kappa\mathbf{a}_3$, we have:

$$\begin{aligned} \mathbf{t}' &= N'\mathbf{a}_1 + T'\mathbf{a}_2 + \mathbf{k}\times\mathbf{t} = \big(N' - \kappa T\big)\,\mathbf{a}_1 + \big(T' + \kappa N\big)\,\mathbf{a}_2 \\ \mathbf{m}' &= M'\mathbf{a}_3 + \mathbf{k}\times\mathbf{m} = M'\mathbf{a}_3 \end{aligned}$$

in which $\mathbf{a}_i = \mathbf{R}\bar{\mathbf{a}}_i$ must be used. Moreover, since, $\mathbf{x}' = \mathbf{R}\,(\bar{\mathbf{a}}_1 + \mathbf{e})$:

$$\mathbf{x}'\times\mathbf{t} = [(1+\varepsilon)\,\mathbf{a}_1 + \gamma\mathbf{a}_2]\times(N\mathbf{a}_1 + T\mathbf{a}_2) = [(1+\varepsilon)\,T - \gamma N]\,\mathbf{a}_3$$

56. Equations [2.213] are of type $\boldsymbol{D}^\star(\boldsymbol{w},\boldsymbol{w}')\,\boldsymbol{\sigma} = \boldsymbol{p}$, where $\boldsymbol{\sigma} := (N,T,M)^T$, $\boldsymbol{p} := (\bar{p}_1,\bar{p}_2,\bar{c})^T$ and:

$$\boldsymbol{D}^\star\big(\boldsymbol{w},\boldsymbol{w}'\big) := \begin{pmatrix} (-\cos\theta)\,\partial_s + \kappa\sin\theta & (\sin\theta + \kappa\cos\theta)\,\partial_s & 0 \\ (-\sin\theta)\,\partial_s - \kappa\cos\theta & (-\cos\theta)\,\partial_s + \kappa\sin\theta & 0 \\ \gamma & -(1+\varepsilon) & -\partial_s \end{pmatrix}$$

is the*equilibrium operator* of the planar beam, which depends on $\boldsymbol{w},\boldsymbol{w}'$, via the strains; it is the adjoint of the kinematic operator $\boldsymbol{D}(\boldsymbol{w},\boldsymbol{w}')$. When it is evaluated at the reference configuration, it reduces to the equilibrium operator of the liner theory, $\boldsymbol{D}_0^\star := \boldsymbol{D}^\star(\mathbf{0},\mathbf{0})$, adjoint of $\boldsymbol{D}_0$:

$$\boldsymbol{D}_0^* := \begin{pmatrix} -\partial_s & 0 & 0 \\ 0 & -\partial_s & 0 \\ 0 & -1 & -\partial_s \end{pmatrix}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2.5.3 The Virtual Power Principle

The Virtual Power equation [2.100], $\mathcal{P}_{ext} = \mathcal{P}_{int}$, specializes as follows for the planar beam, when vectors are represented in the reference basis:

$$\begin{aligned}&\int_{\mathcal{S}} (\bar{p}_1 v_1 + \bar{p}_2 v_2 + \bar{c}\omega)\, ds + \sum_{H=A}^{B} \left(\bar{P}_{1H} v_{1H} + \bar{P}_{2H} v_{2H} + \bar{C}_H \omega_H\right) \\ &\quad = \int_{\mathcal{S}} (N\dot{\varepsilon} + T\dot{\gamma} + M\dot{\kappa})\, ds, \qquad \forall \left(\dot{u}_1, \dot{u}_2, \dot{\theta}\right)\end{aligned} \quad [2.215]$$

This equation can also be used to alternatively derive the scalar balance equations and boundary conditions, according to the balance power approach. Indeed, if the strain-rate-velocity relationships [2.209] are introduced in this equation, and $v_1 = \dot{u}_1$, $v_2 = \dot{u}_2$, $\omega = \dot{\theta}$ are used, equation [2.215] becomes:

$$\begin{aligned}&\int_{\mathcal{S}} \left(\bar{p}_1 \dot{u}_1 + \bar{p}_2 \dot{u}_2 + \bar{c}\dot{\theta}\right) ds + \sum_{H=A}^{B} \left(\bar{P}_{1H} \dot{u}_{1H} + \bar{P}_{2H} \dot{u}_{2H} + \bar{C}_H \dot{\theta}_H\right) \\ &\quad = \int_{\mathcal{S}} \left[N \left(\dot{u}_1' \cos\theta + \dot{u}_2' \sin\theta + \dot{\theta}\gamma\right) ds\right. \\ &\quad \left. + T \left(-\dot{u}_1' \sin\theta + \dot{u}_2' \cos\theta - (1+\varepsilon)\theta\right) + M\dot{\theta}'\right] ds\end{aligned} \quad [2.216]$$

or, after an integration by parts:

$$\begin{aligned}&\int_{\mathcal{S}} \Big\{ \left[(N\cos\theta)' + (T\sin\theta)' + \bar{p}_1\right] \dot{u}_1 \\ &\quad + \left[-(N\sin\theta)' + (T\cos\theta)' + \bar{p}_2\right] \dot{u}_2 \\ &\quad + \left[M' - N\gamma + T(1+\varepsilon) + \bar{c}\right] \dot{\theta} \Big\}\, ds \\ &\quad + \left\{ \left[\pm (N\cos\theta - T\sin\theta) + \bar{P}_1\right] \dot{u}_1 \right\}_H \\ &\quad + \left\{ \left[\pm (N\sin\theta + T\cos\theta) + \bar{P}\right] \dot{u}_2 \right\}_H + \left\{ \pm M\dot{\theta} + \bar{C} \right\}_H = 0\end{aligned} \quad [2.217]$$

Finally, since $(\sin\theta)' = \kappa\cos\theta$, $(\cos\theta)' = -\kappa\sin\theta$, equations [2.213] are re-obtained, together with equation [2.214] on the free boundary.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2.5.4 Constitutive laws

By confining ourselves to the uncoupled linear elastic law, we have:

$$
\begin{aligned}
N &= \mathring{N} + EA\varepsilon \\
T &= \mathring{T} + GA_s\gamma \\
M &= \mathring{M} + EJ\kappa
\end{aligned}
\qquad [2.218]
$$

where $GA_s\colon = GA_2$ is the shear stiffness, $EJ := EJ_3$ is the bending stiffness and quantities with over-ring symbol are prestresses.

If a KV model is adopted, the constitutive law becomes:

$$
\begin{aligned}
N &= \mathring{N} + EA\varepsilon + \eta A\dot{\varepsilon} \\
T &= \mathring{T} + GA_s\gamma + \zeta A_s\dot{\gamma} \\
M &= \mathring{M} + EJ\kappa + \eta J\dot{\kappa}_i
\end{aligned}
\qquad [2.219]
$$

where η, ζ are viscosity coefficients.

2.5.5 The Fundamental Problem

The equations governing the motion (or the equilibrium, in the static case) of the planar beam, are obtained by expressing the balance equations [2.213] and boundary conditions [2.214] in terms of the configuration variables, once the constitutive law (e.g. equation [2.218] and the strain–displacement relationships (equations [2.206]) are substituted into them.

The following exact equations of motion are obtained:

$$
\begin{aligned}
&\{[\mathring{N} + EA((1+u_1')\cos\theta + u_2'\sin\theta - 1)]' - \theta'[\mathring{T} + GA_s(u_2'\cos\theta \\
&\quad - (1+u_1')\sin\theta)]\}\cos\theta - \{[\mathring{T} + GA_s(u_2'\cos\theta - (1+u_1')\sin\theta)]' \\
&\quad + \theta'[\mathring{N} + EA((1+u_1')\cos\theta + u_2'\sin\theta - 1)]\}\sin\theta + \bar{p}_1 - m\ddot{u}_1 = 0 \\
&\{[\mathring{N} + EA((1+u_1')\cos\theta + u_2'\sin\theta - 1)]' - \theta'[\mathring{T} + GA_s(u_2'\cos\theta \\
&\quad - (1+u_1')\sin\theta)]\}\sin\theta + \{[\mathring{T} + GA_s(u_2'\cos\theta - (1+u_1')\sin\theta)]' \\
&\quad + \theta'[\mathring{N} + EA((1+u_1')\cos\theta + u_2'\sin\theta - 1)]\}\cos\theta + \bar{p}_2 - m\ddot{u}_2 = 0 \\
&\{\mathring{M} + EJ\theta'\}' + [(1+u_1')\cos\theta + u_2'\sin\theta][\mathring{T} + GA_s(u_2'\cos\theta \\
&\quad - (1+u_1')\sin\theta)] - [u_2'\cos\theta - (1+u_1')\sin\theta][\mathring{N} + EA((1+u_1')\cos\theta \\
&\quad + u_2'\sin\theta - 1)] + \bar{c} - I\ddot{\theta} = 0
\end{aligned}
\qquad [2.220]
$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with the mechanical boundary conditions:

$$\begin{aligned}
&\mp[[\mathring{N} + EA((1+u_1')\cos\theta + u_2'\sin\theta - 1)]\cos\theta - [\mathring{T} + GA_s(u_2'\cos\theta \\
&\quad - (1+u_1')\sin\theta)]\sin\theta]_H = \bar{P}_{1H} \\
&\mp[[\mathring{N} + EA((1+u_1')\cos\theta + u_2'\sin\theta - 1)]\sin\theta - [\mathring{T} + GA_s(u_2'\cos\theta \\
&\quad - (1+u_1')\sin\theta)]\cos\theta]_H = \bar{P}_{2H} \\
&\mp[\mathring{M} + EJ\theta']_H = \bar{C}_H \qquad\qquad H = A, B
\end{aligned} \qquad [2.221]$$

and the geometric boundary conditions [2.204].

Linearized theory

The linearized version of the strain–displacement relationships [2.206] is:

$$\begin{aligned}
&\varepsilon = u_1' \\
&\gamma = u_2' - \theta \\
&\kappa = \theta'
\end{aligned} \qquad [2.222]$$

Concerning the balance equations [2.213], we split the forces in (large) preloads and (small) incremental contribution, $\bar{p}_i = \mathring{p}_i + \tilde{p}_i$, $\bar{c}_i = \mathring{c}_i + \tilde{c}_i$, as well the stresses in $N = \mathring{N} + \tilde{N}$, $T = \mathring{T} + \tilde{T}$, $M = \mathring{M} + \tilde{M}$, and then we expand up to first-order quantities. By accounting for the pre-existing equilibrium, the balance equations transform into:

$$\begin{aligned}
&\tilde{N}' - \theta'\mathring{T} - \theta\mathring{T}' + \tilde{p}_1 - m\ddot{u}_1 = 0 \\
&\tilde{T}' + \theta\mathring{N}' + \theta'\mathring{N} + \tilde{p}_2 - m\ddot{u}_2 = 0 \\
&\tilde{M}' + (1+u_1')\mathring{T} + \tilde{T} - (u_2' - \theta)\mathring{N} + \tilde{c} - I\ddot{\theta} = 0
\end{aligned} \qquad [2.223]$$

By applying the same procedure to the boundary conditions [2.214], we have:

$$\begin{aligned}
&\mp[\tilde{N} - \mathring{T}\theta]_H = \tilde{P}_{1H} \\
&\mp[\mathring{N}\theta - \tilde{T}]_H = \tilde{P}_{2H} \\
&\mp\tilde{M}_H = \tilde{C}_H
\end{aligned} \qquad [2.224]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Finally, by using the uncoupled linear elastic law [2.218], we obtain the equations of motion[57]:

$$\begin{aligned}
&[EAu_1']' - \theta'\mathring{T} - \theta\mathring{T}' + \tilde{p}_1 - m\ddot{u}_1 = 0 \\
&[GA_s(u_2' - \theta)]' + \theta\mathring{N}' + \theta'\mathring{N} + \tilde{p}_2 - m\ddot{u}_2 = 0 \\
&[EJ\theta']' + GA_s(u_2' - \theta) + u_1'\mathring{T} - (u_2' - \theta)\mathring{N} + \tilde{c} - I\ddot{\theta} = 0
\end{aligned} \qquad [2.225]$$

Similarly, the mechanical boundary conditions are:

$$\begin{aligned}
&\mp [EAu_1' - \theta\mathring{T}]_H = \tilde{P}_{1H} \\
&\mp [GA_s(u_2' - \theta) + \theta\mathring{N}]_H = \tilde{P}_{2H} \\
&\mp [EJ\theta']_H = \tilde{C}_H
\end{aligned} \qquad [2.226]$$

If the prestress is ignored (or it is absent), the equations govern the Timoshenko linear planar beam.

2.6 Summary

Here, we summarize the main results of this chapter. We formulated a model of straight beam with rigid cross-sections as a polar, 1D continuum, embedded in a 3D-space. This is made up of body-points that are able to rotate, in addition to translation; conversely, in dynamics, they are able to exchange couple-stresses, in addition to force-stresses.

We started with kinematics, by defining *translations* and *rotations* of body points, the former being a vector field and the latter being a tensor field. To describe strains, we defined *strain* and *curvature* vectors, of two different types, called "current" (or

57. They are of the type $\boldsymbol{L}\boldsymbol{w} + \boldsymbol{G}\boldsymbol{w} = \tilde{\boldsymbol{p}}$ in the field and $\boldsymbol{\mathcal{L}}_H\boldsymbol{w} + \boldsymbol{\mathcal{G}}_H\boldsymbol{w} = \tilde{\boldsymbol{P}}$ on the boundary (equation [1.72]), where:

$$\boldsymbol{L} := \begin{bmatrix} -EA\partial_s^2 & 0 & 0 \\ 0 & -GA_s\partial_s^2 & GA_s\partial_s \\ 0 & -GA_s\partial_s & GA_s - EJ\partial_s^2 \end{bmatrix}, \qquad \boldsymbol{G} := \begin{bmatrix} 0 & 0 & \mathring{T}' + \mathring{T}\partial_s \\ 0 & 0 & -\mathring{N}' - \mathring{N}\partial_s \\ -\mathring{T}\partial_s & \mathring{N}\partial_s & -\mathring{N} \end{bmatrix}$$

are elastic and geometric stiffness operators, and:

$$\boldsymbol{\mathcal{L}}_H := \mp \begin{bmatrix} EA\partial_s & 0 & 0 \\ 0 & GA_s\partial_s & -GA_s \\ 0 & 0 & EJ\partial_s \end{bmatrix}_H, \qquad \boldsymbol{\mathcal{G}}_H := \mp \begin{bmatrix} 0 & 0 & -\mathring{T} \\ 0 & 0 & \mathring{N} \\ 0 & 0 & 0 \end{bmatrix}_H$$

are their counterpart on the free boundary.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

left), if they follow the local rotation, or called "reference" (or right), if they precede the rotation. Current and reference strains have the same scalar components in the current and reference bases, respectively. Although the current measures have a clearer geometrical meaning, they suffer from the drawback that not only their components, but also the unit vectors of the current basis depend on time. By accounting for time-rates, we introduced *velocity* and *spin* fields, both of which are vector fields, since the latter is described by an antisymmetric tensor. In addition, we defined the *stretching velocity gradients*, as that part of the material gradient of the velocity which is *not* a mere rotation. While the reference *strain-rates* equate the stretching velocity gradients, the current strain-rates do not. However, if reference is made to triplets of components (taken in the proper bases), instead of to vectors or tensors, no such problem arises (since unit vectors disappear).

After that, we addressed the dynamics of beams. We first presented the *power balance approach* based on the VPP. In this framework, the stresses are introduced as the dual quantities of the stretching velocity gradients. The principle leads to (a) balance equations in the field, which are differential equations involving stresses and external forces, of active and inertial type, as well (b) boundary conditions, from which natural (or mechanical) conditions spring on the portion of the boundary that is unconstrained. Second, we discussed an alternative *force balance approach* based on the linear and angular *momentum principles*, valid for rigid body dynamics, and accepted here as postulates. The procedure requires primarily defining the stresses as contact interactions between adjacent body-points made up of a force-stress and a couple-stress vector. The "Principle of action and reaction" has also to be invoked. Boundary conditions are enforced on the free boundary by requiring that the emerging stresses equate the applied external forces. The two procedures supply the same equations. We projected them onto the current, as well onto the reference bases. A third form of the scalar equations was also derived, in which the balance of the forces is expressed in the reference basis, while the balance of the moments is enforced in the *spin basis*, i.e. the non-orthogonal basis formed by the directions along which the spin vector naturally decomposes in elementary contributions of moduli $\dot{\theta}_i$. These moment equations naturally appear in a *Lagrangian approach* in which the virtual motion is expressed by the time-derivatives of the Tait–Bryan angles $\dot{\theta}_i$, instead of the components of the spin vector $\bar{\omega}_i$.

To complete the modeling, we discussed constitutive laws, mainly confining ourselves to the hyperelastic behavior. This called for postulating the existence of an *elastic potential*, function of the strains, from which stresses are derived via differentiation (Green law). We paid special attention to a *linear* (or Hooke) law by accounting for possible prestress acting in the reference configuration, assumed to be known. We showed how to identify the constants of the elastic potential from a 3D-model of beam, after suitable kinematic hypotheses had been introduced. The procedure led us to formulate a linear *diagonal* law, in which the usual axial, shear, torsional and bending stiffness of the elementary theory of beam appear. However, it

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

was mentioned that such a simple model, although commonly used, is unable to account for the elongation of the longitudinal fibers of beam when a large twist occurs. Therefore, we modified the model, although in an inconsistent way, to account for this phenomenon, which produces a *nonlinear coupling* between torsion, on the one side, and extension and bending, on the other side. Moreover, we also suggested how to formulate an elastic law for *beam-like structures*, where a homogenization process provides equivalent elastic coefficients for the beam model. Finally, we brifely mentioned linear viscoelastic laws, based on elementary rheological models, such as the KV and the standard models.

All the equations of the problem were combined to formulate the Fundamental Problem for the straight beam, governed by 18 scalar equations in as many unknowns. Linearized equations, accounting for large prestresses, were also derived.

In conclusion, the whole model was revisited to specialize it for the case of *planar beams*, very frequent in applications. The strong simplification of this model relies on the fact that the rotation of the cross-section occurs around a fixed axis, orthogonal to the plane. Accordingly, the curvature vector and the spin vector are also orthogonal to this plane, as well as the couple-stress and the angular acceleration. In contrast, the displacement, velocity and force-stress are vectors contained in the plane of the beam. The exact equations of motion for prestressed beams, together with their linearized version, were supplied.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Chapter 3

Curved Beams

We consider a curved beam, of arbitrary shape, immersed in a three-dimensional (3D)-space and model it as a one-dimensional polar continuum. The reference configuration is described by the parametric equations of the centerline and by a deviation angle able to identify the attitude of the local principal inertia basis. Initial curvature tensor and vector are consistently introduced. While the strain vector is defined as for the straight beam, a new "change of curvature" vector is defined, measuring the variation of the curvature undergone by the beam in passing from the reference to the current configuration. Vector balance equations are derived by the Virtual Power Principle (VPP), and then projected onto the reference basis. An elastic constitutive law is accounted for. The Fundamental Problem is then formulated and matrix as well as extended expressions for the exact equations of motion are given for unprestressed elastic beams. The linearized equations of motion for elastic prestressed beams are also worked out. Finally, the special case of *planar beam* is analyzed, for which explicit expressions of the equations of motion are derived.

3.1 The reference configuration and the initial curvature

We consider a curved beam, made up of a flexible centerline and rigid cross-sections, and we consider it as a one-dimensional polar continuum endowed with a local rigid structure. The centerline, in the reference configuration, is a regular (generally not planar) curve $\mathcal{S}$, of parametric equations $\bar{\mathbf{x}} = \bar{\mathbf{x}}(s)$, where s is the arclength and $\bar{\mathbf{x}}$ is a vector describing the position of the generic material point P

Mathematical Models of Beams and Cables Angelo Luongo and Daniele Zulli
© 2013 ISTE Ltd. Published 2013 by ISTE Ltd.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with respect to an arbitrary pole O (Figure 3.1(a)). Denoting by $\mathcal{B}_e := (\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ a right orthogonal *external* triad, it is $\bar{\mathbf{x}} = \sum_{j=1}^{3} \bar{x}_j(s)\mathbf{i}_j$.

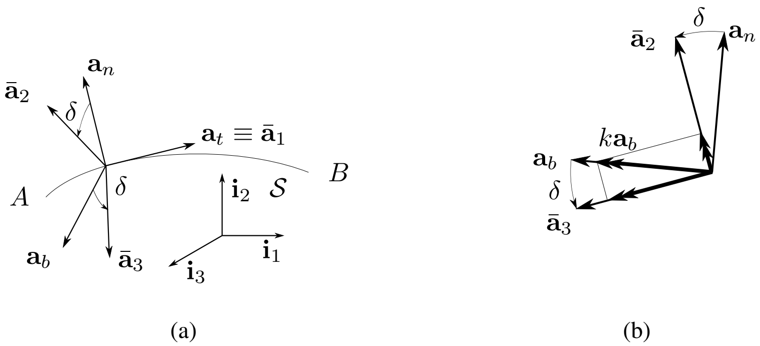


Figure 3.1: Curved beam in the reference configuration: (a) triad $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ external to the beam, triad $(\mathbf{a}_t, \mathbf{a}_n, \mathbf{a}_b)$ intrinsic to the centerline and triad $(\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3)$ intrinsic to the cross-section; and (b) δ deviation angle and projection of the vector $k\mathbf{a}_b$.

Once the centerline has been described, we have to specify the attitude of the cross-sections as a function of s, namely the orientation of a triad of directors $\bar{\mathcal{B}} := (\bar{\mathbf{a}}_1(s), \bar{\mathbf{a}}_2(s), \bar{\mathbf{a}}_3(s))$ attached to them. We already noted that, in order to simplify the constitutive law and inertia forces, it is convenient to choose $\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3$ as *principal axes of inertia* of the cross-section, $\bar{\mathbf{a}}_1$ being orthogonal to them, and therefore tangent to $\mathcal{S}$; hence, the task is to fix these axes with respect to the $\mathcal{B}_e$ triad. The best strategy is the following: (a) first, by using the parametric equations for the centerline and the Frenet formulas (equations [2.33], find the TNB (or Frenet) basis $\mathcal{B}_f := (\mathbf{a}_t(s), \mathbf{a}_n(s), \mathbf{a}_b(s))$, which are intrinsic to the curve; (b) then, define the *angle of deviation* $\delta(s) := \arccos(\mathbf{a}_n \cdot \bar{\mathbf{a}}_2) \equiv \arccos(\mathbf{a}_b \cdot \bar{\mathbf{a}}_3)$ between the inertia basis $\bar{\mathcal{B}}$ and the TNB basis $\mathcal{B}_f$ (Figure 3.1(b)).

Concerning task (a), we have:

$$\mathbf{a}_t = \bar{\mathbf{x}}', \qquad \mathbf{a}_n = \frac{1}{k}\bar{\mathbf{x}}'', \qquad \mathbf{a}_b = \frac{1}{k}(\bar{\mathbf{x}}' \times \bar{\mathbf{x}}'') \tag{3.1}$$

with the *curvature and torsion of the curve* $\bar{\mathbf{x}}(s)$ given by:

$$k := \|\bar{\mathbf{x}}''\|, \qquad \tau := \frac{\bar{\mathbf{x}}' \times \bar{\mathbf{x}}'' \cdot \bar{\mathbf{x}}'''}{\|\bar{\mathbf{x}}''\|^2} \tag{3.2}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

or by using components:

$$k=\sqrt{\bar{x}_1''^2+\bar{x}_3''^2+\bar{x}_3''^2},\qquad \tau=\frac{1}{\bar{x}_1''^2+\bar{x}_3''^2+\bar{x}_3''^2}\begin{vmatrix}\bar{x}_1' & \bar{x}_2' & \bar{x}_3'\\ \bar{x}_1'' & \bar{x}_2'' & \bar{x}_3''\\ \bar{x}_1''' & \bar{x}_2''' & \bar{x}_3'''\end{vmatrix} \qquad [3.3]$$

Indeed, $\bar{\mathbf{x}}'$ is the unit tangent vector; since $\mathbf{a}_t'=\bar{\mathbf{x}}''=\|\bar{\mathbf{x}}''\|\text{vers}\,(\bar{\mathbf{x}}'')$, by accounting for equation [2.33a], k and $\mathbf{a}_n$ are derived; hence, $\mathbf{a}_b=\mathbf{a}_t\times\mathbf{a}_n$ is used; finally, the torsion is evaluated as $\tau=\mathbf{a}_b\cdot\mathbf{a}_n'$ (see equation [2.35]).

Concerning task (b), the inertia triad is obtained by rotating the Frenet triad by an angle $\delta(s)$ around $\mathbf{a}_t$; consequently:

$$\begin{aligned}\bar{\mathbf{a}}_1&=\mathbf{a}_t\\ \bar{\mathbf{a}}_2&=\mathbf{a}_n\cos\delta+\mathbf{a}_b\sin\delta\\ \bar{\mathbf{a}}_3&=-\mathbf{a}_n\sin\delta+\mathbf{a}_b\cos\delta\end{aligned} \qquad [3.4]$$

By differentiating equation [3.4], and using the Frenet formulas [2.33] and the inverse transformation of equation [3.4], the derivatives $\bar{\mathbf{a}}_j'$ are obtained in terms of the same $\bar{\mathbf{a}}_j$'s, as follows:

$$\begin{aligned}\bar{\mathbf{a}}_1'&=\bar{\mathbf{a}}_2k\cos\delta-\bar{\mathbf{a}}_3k\sin\delta\\ \bar{\mathbf{a}}_2'&=-\bar{\mathbf{a}}_1k\cos\delta+\bar{\mathbf{a}}_3(\tau+\delta')\\ \bar{\mathbf{a}}_3'&=\bar{\mathbf{a}}_1k\sin\delta-\bar{\mathbf{a}}_2(\tau+\delta')\end{aligned} \qquad [3.5]$$

These equations suggest defining, in analogy with equations [2.38] and [2.41], an *initial curvature skew tensor* $\bar{\mathbf{K}}$, or, equivalently, an *initial curvature vector* $\bar{\mathbf{k}}$, such that:

$$\bar{\mathbf{a}}_j'=\bar{\mathbf{K}}\bar{\mathbf{a}}_j=\bar{\mathbf{k}}\times\bar{\mathbf{a}}_j \qquad [3.6]$$

whose scalar representation in $\bar{\mathcal{B}}$ is:

$$\bar{\mathbf{k}}=\bar{\kappa}_1\bar{\mathbf{a}}_1+\bar{\kappa}_2\bar{\mathbf{a}}_2+\bar{\kappa}_3\bar{\mathbf{a}}_3 \qquad [3.7]$$

Substitution of equation [3.7] into equation [3.6] and comparison with equation [3.5] yields the components:

$$\bar{\kappa}_1:=(\tau+\delta'),\qquad \bar{\kappa}_2:=k\sin\delta,\qquad \bar{\kappa}_3:=k\cos\delta \qquad [3.8]$$

In conclusion, the effect of the deviation angle δ only consists of adding an increment δ' to the torsion of the curve. Indeed, the curvature component $k\mathbf{a}_b$ remains unaltered, but it is now projected onto the $\bar{\mathcal{B}}$-basis, and not onto the TNB basis, and therefore it provides two non-zero components on $\bar{\mathbf{a}}_2,\bar{\mathbf{a}}_3$ (see Figure 3.1(b)).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

REMARK 3.1. Equation [3.6] permits us to evaluate the space-derivative of a generic *vector* attached to the basis $\bar{\mathcal{B}}$, e.g. $\mathbf{w} = \sum_{i=1}^{3} w_i \bar{\mathbf{a}}_i$. Since $\mathbf{w}' = \sum_{i=1}^{3} \left(w_i' \bar{\mathbf{a}}_i + w_i \bar{\mathbf{a}}_i' \right)$, it follows that:

$$\mathbf{w}' = \sum_{i=1}^{3} w_i' \bar{\mathbf{a}}_i + \bar{\mathbf{k}} \times \mathbf{w} \qquad [3.9]$$

which is known as *Poisson formula*. When this is represented in $\bar{\mathcal{B}}$, it yields:

$$[\boldsymbol{w}']_{\bar{\mathcal{B}}} = [\boldsymbol{w}]'_{\bar{\mathcal{B}}} + \bar{\boldsymbol{K}} \boldsymbol{w} \qquad [3.10]$$

where $[\boldsymbol{w}]'_{\bar{\mathcal{B}}}$ is the column matrix of the derivatives of the components of $\mathbf{w}$ in $\bar{\mathcal{B}}$.

REMARK 3.2. Equation [3.6] also permits us to evaluate the space-derivative of a generic *tensor* attached to $\bar{\mathcal{B}}$, which we will use further. By letting:

$$\mathbf{W} = \sum_{i=1}^{3} \sum_{j=1}^{3} w_{ij} \bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j \qquad [3.11]$$

where the symbol $\otimes$ denotes the tensor (or dyadic) product; after differentiation of both members, it follows:

$$\begin{aligned} \mathbf{W}' &= \sum_{i=1}^{3} \sum_{j=1}^{3} \left(w_{ij}' \bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j + w_{ij} \bar{\mathbf{a}}_i' \otimes \bar{\mathbf{a}}_j + w_{ij} \bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j' \right) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \left(w_{ij}' \bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j + \bar{\mathbf{K}} w_{ij} \bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j + w_{ij} \bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j \bar{\mathbf{K}}^T \right) \end{aligned} \qquad [3.12]$$

where equation [3.6] has been used, and a known property is taken into account[1]. Equation [3.12] is the tensor counterpart of equation [3.9], valid for a vector. Therefore, the scalar representation of $\mathbf{W}'$ in $\bar{\mathcal{B}}$ is:

$$\left[\boldsymbol{W}' \right]_{\bar{\mathcal{B}}} = [\boldsymbol{W}]'_{\bar{\mathcal{B}}} + \bar{\boldsymbol{K}} \boldsymbol{W} - \boldsymbol{W} \bar{\boldsymbol{K}} \qquad [3.13]$$

where $\bar{\boldsymbol{K}}^T = -\bar{\boldsymbol{K}}$ has been accounted for; here, $[\boldsymbol{W}]'_{\bar{\mathcal{B}}}$ is the matrix of the derivatives of the components of $\mathbf{W}$ in $\bar{\mathcal{B}}$ and the last two terms account for the derivatives of the unit vectors of the basis.

---

1. The dyadic product between two vectors $\mathbf{u}$ and $\mathbf{v}$ is a second-order tensor, $\mathbf{T} := \mathbf{u} \otimes \mathbf{v}$ such that $\mathbf{T}\mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$, $\forall \mathbf{w}$. Therefore, $\mathbf{u} \otimes \mathbf{L}\mathbf{v} = (\mathbf{u} \otimes \mathbf{v})\, \mathbf{L}^T$ holds, where $\mathbf{L}$ is a generic second-order tensor, since $(\mathbf{u} \otimes \mathbf{L}\mathbf{v})\, \mathbf{w} = (\mathbf{L}\mathbf{v} \cdot \mathbf{w})\, \mathbf{u} = \left(\mathbf{v} \cdot \mathbf{L}^T \mathbf{w} \right) \mathbf{u} = (\mathbf{u} \otimes \mathbf{v})\, \mathbf{L}^T \mathbf{w}$. The scalar representation of the tensor in a selected basis is a matrix $\boldsymbol{T} = \boldsymbol{u}\boldsymbol{v}^T$, where $\boldsymbol{u}, \boldsymbol{v}$ are column matrices, representing $\mathbf{u}, \mathbf{v}$ in the same basis. As a particular case, $\bar{\mathbf{A}}_{ij} := \bar{\mathbf{a}}_i \otimes \bar{\mathbf{a}}_j$ is a tensor, whose scalar representation is a matrix of zeros, except for the (i, j)-entry, which is equal to 1; therefore $\left(\bar{\mathbf{A}}_{ij} \right)$, for $i, j = 1, 2, 3$ is a basis for the second-order tensor space.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

3.2 The beam model in the 3D-space

In studying kinematics, dynamics and constitutive laws of curved beams, we will closely follow the analysis carried out for straight beams, mainly focusing our attention on the differences due to the existence of an initial curvature.

3.2.1 Kinematics

The displacement and rotation fields

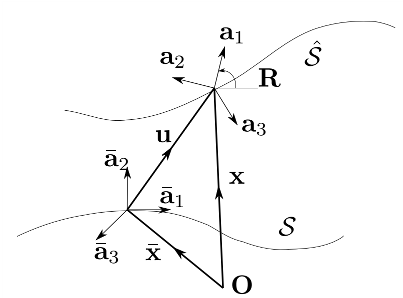


Figure 3.2: Displacement and rotation.

Let us consider the beam in the current configuration occupied at the time t, in which the centerline lies on a line $\hat{\mathcal{S}}$, while the cross-sections are no longer orthogonal to the centerline (Figure 3.2). Let $\mathbf{x}$ be the positions of the material point P in the current configurations and let:

$$\mathbf{u} := \mathbf{x}(s,t) - \bar{\mathbf{x}}(s) \tag{3.14}$$

be the displacement of P at the time t. By introducing *intrinsic scalar components* in the basis $\bar{\mathcal{B}}$, we have:

$$\mathbf{u} = u_1(s,t)\bar{\mathbf{a}}_1(s) + u_2(s,t)\bar{\mathbf{a}}_2(s) + u_3(s,t)\bar{\mathbf{a}}_3(s) \tag{3.15}$$

In the current configuration, the triad of directors forms a basis $\mathcal{B} := (\mathbf{a}_1(s,t), \mathbf{a}_2(s,t), \mathbf{a}_3(s,t))$, which is related to $\bar{\mathcal{B}}$ by a rotation:

$$\mathbf{a}_j(s,t) = \mathbf{R}(s,t)\bar{\mathbf{a}}_j(s) \tag{3.16}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The proper orthogonal tensor $\mathbf{R}$ is represented in $\bar{\mathcal{B}}$ by the matrix (equation [2.6]):

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta_2\cos\theta_3 & \sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 & \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 \\ \cos\theta_2\sin\theta_3 & \sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 \\ -\sin\theta_2 & \sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 \end{bmatrix} \quad [3.17]$$

The geometric boundary conditions

Displacement and rotation fields must satisfy the prescribed geometric boundary conditions [2.17] at the constrained ends of the beam (if any), i.e:

$$\mathbf{u}_H = \breve{\mathbf{u}}_H(t), \qquad \mathbf{R}_H = \breve{\mathbf{R}}_H(t) \qquad H = A, B \quad [3.18]$$

or, in equivalent matrix form:

$$\boldsymbol{u}_H = \breve{\boldsymbol{u}}_H(t), \qquad \boldsymbol{\theta}_H = \breve{\boldsymbol{\theta}}_H(t) \quad [3.19]$$

The strain vector

The (reference) strain vector is defined as for the straight beam (equation [2.24]), i.e.:

$$\mathbf{e} := \mathbf{R}^T\mathbf{x}' - \bar{\mathbf{a}}_1 \quad [3.20]$$

and its representation in $\bar{\mathcal{B}}$ is:

$$\mathbf{e} = \varepsilon\bar{\mathbf{a}}_1 + \gamma_2\bar{\mathbf{a}}_2 + \gamma_3\bar{\mathbf{a}}_3 \quad [3.21]$$

Curvature tensors and vectors

Concerning the curvature, we have to account for the initial contribution, which was absent in the straight beam. We start by defining a current $\mathbf{K}_c$, and a reference $\mathbf{K}_r$ curvature tensors, as in equations [2.38] and [2.42]:

$$\begin{aligned} \mathbf{a}'_j &= \mathbf{K}_c\mathbf{a}_j \\ \mathbf{R}^T\mathbf{a}'_j &= \mathbf{K}_r\bar{\mathbf{a}}_j \end{aligned} \quad [3.22]$$

As for the straight beam (equation [2.45]), $\mathbf{K}_r = \mathbf{R}^T\mathbf{K}_c\mathbf{R}$ holds; hence, the two tensors have the same components in $\mathcal{B}$ and $\bar{\mathcal{B}}$, respectively.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

To evaluate $\mathbf{a}_j'$ we need to differentiate equation [3.16], now accounting for the space dependence of the unit vectors, i.e.:

$$\mathbf{a}_j' = \mathbf{R}'\bar{\mathbf{a}}_j + \mathbf{R}\bar{\mathbf{a}}_j' = \left(\mathbf{R}' + \mathbf{R}\bar{\mathbf{K}}\right)\bar{\mathbf{a}}_j \qquad [3.23]$$

where equation [3.6] has been used. By substituting this result in equation [3.22] and using again equation [3.16], the two curvature tensors are obtained:

$$\begin{aligned} \mathbf{K}_c &= \mathbf{R}'\mathbf{R}^T + \mathbf{R}\bar{\mathbf{K}}\mathbf{R}^T \\ \mathbf{K}_r &= \mathbf{R}^T\mathbf{R}' + \bar{\mathbf{K}} \end{aligned} \qquad [3.24]$$

in each of which an additional term appears, with respect to the definitions [2.40] and [2.43]. Both the tensors are skew-symmetric, being $\bar{\mathbf{K}}$ is skew-symmetric, too.

The tensors introduced, however, measure the present state of the beam, but they are *not* a measure of strain, since the beam was initially curve. Therefore, we need to define a new quantity that is able to account for the *increment of curvature*. Since the initial curvature refers to the reference configuration, it is suitable to use the reference curvature tensor $\mathbf{K}_r$ and to define[2]:

$$\mathbf{X} := \mathbf{K}_r - \bar{\mathbf{K}} \qquad [3.25]$$

as the *change of curvature tensor*, therefore:

$$\mathbf{X} = \mathbf{R}^T\mathbf{R}' \qquad [3.26]$$

The previous tensors, $\mathbf{K}_c$, $\mathbf{K}_r$, $\mathbf{X}$ being all skew-symmetric, are equivalent to their associated axial vectors, $\mathbf{k}_c$, $\mathbf{k}_r$, $\boldsymbol{\chi}$, respectively. By defining:

$$\begin{aligned} \mathbf{a}_j' &= \mathbf{k}_c \times \mathbf{a}_j \\ \mathbf{R}^T\mathbf{a}_j' &= \mathbf{k}_r \times \bar{\mathbf{a}}_j \end{aligned} \qquad [3.27]$$

it follows that $\mathbf{k}_c = \mathbf{R}\mathbf{k}_r$[3]; moreover, the *change of curvature vector* turns out to be[4]:

$$\boldsymbol{\chi} := \mathbf{k}_r - \bar{\mathbf{k}} \qquad [3.28]$$

which admits in $\bar{\mathcal{B}}$ the following scalar representation:

$$\boldsymbol{\chi} := \chi_1\bar{\mathbf{a}}_1 + \chi_2\bar{\mathbf{a}}_2 + \chi_3\bar{\mathbf{a}}_3 \qquad [3.29]$$

Since $\mathbf{k}_r = \sum_{i=1}^3 \kappa_i\bar{\mathbf{a}}_i$ and $\bar{\mathbf{k}} = \sum_{i=1}^3 \bar{\kappa}_i\bar{\mathbf{a}}_i$, then:

$$\chi_i = \kappa_i - \bar{\kappa}_i, \qquad i = 1, \ldots, 3 \qquad [3.30]$$

As for the straight beam, index r will be discussed further.

2. Here, $\mathbf{X}$ should *not* be read "capital ex", but rather "capital chi".

3. Multiply equation [3.27-b] by $\mathbf{R}$ and observe that $\mathbf{R}\left(\mathbf{k}_r \times \bar{\mathbf{a}}_j\right) = \mathbf{R}\mathbf{k}_r \times \mathbf{R}\bar{\mathbf{a}}_j = \mathbf{R}\mathbf{k}_r \times \mathbf{a}_j$.

4. Indeed, from equation [3.25], $\mathbf{X}\bar{\mathbf{a}}_j = \left(\mathbf{K}_r - \bar{\mathbf{K}}\right)\bar{\mathbf{a}}_j$, from which $\boldsymbol{\chi} \times \bar{\mathbf{a}}_j = \left(\mathbf{k}_r - \bar{\mathbf{k}}\right) \times \bar{\mathbf{a}}_j$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

REMARK 3.3. Formula [3.26] is surprisingly similar to equation [2.43], valid for the straight beam, and, at a first glance, it seems that the contribution of the initial curvature disappeared. However, we have to keep in mind that we are now dealing with a curvilinear coordinate s, so that $\mathbf{R}'$ *does account for the initial curvature*, according to equation [3.12].

REMARK 3.4. The change of curvature can also be written in terms of the current curvature vector as $\boldsymbol{\chi} := \mathbf{R}^T\mathbf{k}_c - \bar{\mathbf{k}}$. It is interesting to observe that this equation has the same form as equation [3.20], which is relevant to strain, and, indeed, it keeps the same meaning. As a matter of fact, the current curvature of the beam, $\mathbf{k}_c$, must first be pulled back to the reference configuration before it can be compared with the initial curvature; their difference is a measure of strain.

The strain–displacement relationships

Since $\mathbf{x}' = \bar{\mathbf{a}}_1 + \mathbf{u}'$, equation [3.20] also becomes $\mathbf{e} = \mathbf{R}^T(\bar{\mathbf{a}}_1 + \mathbf{u}') - \bar{\mathbf{a}}_1$. To represent it in $\bar{\mathcal{B}}$, the Poisson formula [3.10] must be used to express the components of $\mathbf{u}'$. Therefore, in matrix form, we have (compare it with equation [2.51]):

$$\boldsymbol{e} = \boldsymbol{R}^T\left(\bar{\boldsymbol{a}}_1 + \boldsymbol{u}' + \bar{\boldsymbol{K}}\boldsymbol{u}\right) - \bar{\boldsymbol{a}}_1 \qquad [3.31]$$

or, in extended form:

$$\begin{pmatrix}\varepsilon\\ \gamma_2\\ \gamma_3\end{pmatrix} = \boldsymbol{R}^T\left(\begin{pmatrix}1+u_1'\\ u_2'\\ u_3'\end{pmatrix} + \begin{bmatrix}0 & -\bar{\kappa}_3 & \bar{\kappa}_2\\ \bar{\kappa}_3 & 0 & -\bar{\kappa}_1\\ -\bar{\kappa}_2 & \bar{\kappa}_1 & 0\end{bmatrix}\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}\right) - \begin{pmatrix}1\\ 0\\ 0\end{pmatrix} \qquad [3.32]$$

To evaluate the change of curvature components, we represent equation [3.26] in the reference basis. By using equation [3.13] to express the components of $\mathbf{R}'$, we have:

$$\begin{aligned}\boldsymbol{X} &= \boldsymbol{R}^T\left(\boldsymbol{R}' + \bar{\boldsymbol{K}}\boldsymbol{R} - \boldsymbol{R}\bar{\boldsymbol{K}}\right)\\ &= \boldsymbol{R}^T\boldsymbol{R}' + \boldsymbol{R}^T\bar{\boldsymbol{K}}\boldsymbol{R} - \bar{\boldsymbol{K}}\end{aligned} \qquad [3.33]$$

where we adopted the symbol $\boldsymbol{R}'$ instead of the cumbersome $[\boldsymbol{R}]'_{\bar{\mathcal{B}}}$. By performing calculations, and then extracting the components of the axial vector, we obtain (compare it with equation [2.53])[5]:

$$\chi = \boldsymbol{B}_\omega\boldsymbol{\theta}' + \left(\boldsymbol{R}^T - \boldsymbol{I}\right)\bar{\boldsymbol{k}} \qquad [3.34]$$

5. Equation [3.34] can be read as follows: the first term on the right-hand side of equation, $\boldsymbol{B}_\omega\boldsymbol{\theta}'$, is identical to that in equation [2.54], holding for the straight beam; the third term, $\bar{\boldsymbol{k}}$, collects the components of the axial vector of $\bar{\mathbf{K}}$; the second term, $\boldsymbol{R}^T\bar{\boldsymbol{k}}$, collects the components of the axial vector of the rotated tensor $\mathbf{R}^T\bar{\mathbf{K}}\mathbf{R}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where the *spin-basis matrix* $\boldsymbol{B}_\omega$ is defined by equation [2.74] and $\boldsymbol{\theta} := (\theta_1, \theta_2, \theta_3)^T$ is the *rotation pseudo-vector*. In extended form, the previous equation becomes:

$$\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix} = \begin{pmatrix}1 & 0 & -\sin\theta_2\\ 0 & \cos\theta_1 & \sin\theta_1\cos\theta_2\\ 0 & -\sin\theta_1 & \cos\theta_1\cos\theta_2\end{pmatrix}\begin{pmatrix}\theta_1'\\ \theta_2'\\ \theta_3'\end{pmatrix}$$
$$+\begin{bmatrix}\cos\theta_2\cos\theta_3 - 1 & \cos\theta_2\sin\theta_3 & -\sin\theta_2\\ \begin{matrix}\sin\theta_1\sin\theta_2\cos\theta_3\\ -\cos\theta_1\sin\theta_3\end{matrix} & \begin{matrix}\sin\theta_1\sin\theta_2\sin\theta_3\\ +\cos\theta_1\cos\theta_3 - 1\end{matrix} & \sin\theta_1\cos\theta_2\\ \begin{matrix}\cos\theta_1\sin\theta_2\cos\theta_3\\ +\sin\theta_1\sin\theta_3\end{matrix} & \begin{matrix}\cos\theta_1\sin\theta_2\sin\theta_3\\ -\sin\theta_1\cos\theta_3\end{matrix} & \cos\theta_1\cos\theta_2 - 1\end{bmatrix}\begin{pmatrix}\bar{\kappa}_1\\ \bar{\kappa}_2\\ \bar{\kappa}_3\end{pmatrix} \quad [3.35]$$

Velocity and spin

Kinetic quantities for curved beams assume the same expressions as those for straight beams. Indeed, while $\bar{\mathbf{a}}_j$ depends on space (this entailing, as we saw, the occurrence of new terms in strain and curvature), it is independent of time. As an example, $\dot{\mathbf{a}}_j = (\dot{\mathbf{R}}\bar{\mathbf{a}}_j + \mathbf{R}\dot{\bar{\mathbf{a}}}_j) = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{a}_j$, from which $\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T$ follows, as in equation [2.58]. Therefore, the analogy we stressed for straight beams, holding between curvature and spin (tensors and vectors), no longer holds for curved beams. Here, we discuss the most important relationships that we derived in Chapter 2.

Velocity, when expressed in the reference basis, becomes:

$$\mathbf{v} := \dot{\mathbf{u}}(s,t) = \dot{u}_1(s,t)\bar{\mathbf{a}}_1 + \dot{u}_2(s,t)\bar{\mathbf{a}}_2 + \dot{u}_3(s,t)\bar{\mathbf{a}}_3 \quad [3.36]$$

The spin vector and tensor are defined via:

$$\dot{\mathbf{a}}_j = \mathbf{W}\mathbf{a}_j = \boldsymbol{\omega} \times \mathbf{a}_j \quad [3.37]$$

where:

$$\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T \quad [3.38]$$

The components of the spin vector $\boldsymbol{\omega}$, when expressed in the reference or current bases, respectively, become (equation [2.71]):

$$\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{B}}_\omega\dot{\boldsymbol{\theta}}, \qquad \boldsymbol{\omega} = \boldsymbol{B}_\omega\dot{\boldsymbol{\theta}} \quad [3.39]$$

with $\bar{\boldsymbol{B}}_\omega$,$\boldsymbol{B}_\omega$ defined by equations [2.73] and [2.74]. Similarly, the first and second time-derivatives of a vector attached to $\mathcal{B}$ are expressed by equations [2.64] and [2.65].

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

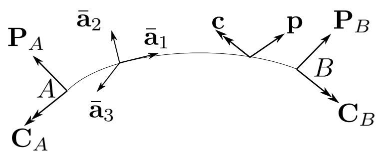


Figure 3.3: Curved beam loaded by forces and couples, both distributed in the body and applied at the ends.

Stretching velocity gradients and strain rates

The stretching velocity and the spin velocity gradients are (equations [2.80] and [2.83]):

$$\begin{aligned} \mathbf{d} &:= \mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}' = \mathbf{R}\dot{\mathbf{e}} \\ \mathbf{s} &:= \boldsymbol{\omega}' = \mathbf{R}\dot{\boldsymbol{\chi}} \end{aligned} \qquad [3.40]$$

where, according to equation [3.28], $\dot{\mathbf{k}}$ has been substituted by $\dot{\boldsymbol{\chi}}$, since $\dot{\bar{\mathbf{k}}} = \mathbf{0}$. In equation [3.40], $\dot{\mathbf{e}} = \dot{\varepsilon}\bar{\mathbf{a}}_1 + \dot{\gamma}_2\bar{\mathbf{a}}_2 + \dot{\gamma}_3\bar{\mathbf{a}}_3$ and $\dot{\boldsymbol{\chi}} = \dot{\chi}_1\bar{\mathbf{a}}_1 + \dot{\chi}_2\bar{\mathbf{a}}_2 + \dot{\chi}_3\bar{\mathbf{a}}_3$ are the strain-rate and the curvature-rate vectors[6].

3.2.2 Dynamics

The dynamic aspects of the problem are now addressed. Results obtained in Chapter 2 are recalled and adapted to the curved beam.

The Virtual Power Principle

Let us consider a curved beam under external forces and couples of density $\mathbf{p}(s,t)$ and $\mathbf{c}(s,t)$, respectively, distributed along the centerline, as well as forces and couples $\mathbf{P}_H(t)$ and $\mathbf{C}_H(t)$, respectively, acting at the ends $H = A, B$ (Figure 3.3).

6. With the notation of the metamodel (Chapter 1), strain rates can be written as $\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}(\boldsymbol{w}, \boldsymbol{w}')\,\dot{\boldsymbol{w}}$, where $\boldsymbol{\varepsilon} := (\varepsilon, \gamma_2, \gamma_3, \chi_1, \chi_3, \chi_3)^T$, $\boldsymbol{w} := (u_1, u_2, u_3, \theta_1, \theta_2, \theta_3)^T$ and $\boldsymbol{D}(\boldsymbol{w}, \boldsymbol{w}')$ is the 6×6 *kinematic operator* of the curved beam.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The external and internal virtual powers become, as in equations [2.89] and [2.90]:

$$\mathcal{P}_{ext} := \int_{\mathcal{S}} (\mathbf{p}\cdot\mathbf{v} + \mathbf{c}\cdot\boldsymbol{\omega})\, ds + \sum_{H=A}^{B} (\mathbf{P}_H\cdot\mathbf{v}_H + \mathbf{C}_H\cdot\boldsymbol{\omega}_H)$$
$$\mathcal{P}_{int} := \int_{\mathcal{S}} [\mathbf{t}\cdot(\mathbf{v}' - \boldsymbol{\omega}\times\mathbf{x}') + \mathbf{m}\cdot\boldsymbol{\omega}']\, ds = \int_{\mathcal{S}} [\mathbf{t}\cdot\mathbf{R}\dot{\mathbf{e}} + \mathbf{m}\cdot\mathbf{R}\dot{\boldsymbol{\chi}}]\, ds \quad [3.41]$$

where equation [3.40] has been used. In equation [3.41], $\mathbf{t}(s,t)$ and $\mathbf{m}(s,t)$ are the force-stress and the couple-stress, admitting the following representation in the current basis $\mathcal{B}$:

$$\mathbf{t} = N\mathbf{a}_1 + T_2\mathbf{a}_2 + T_3\mathbf{a}_3$$
$$\mathbf{m} = M_1\mathbf{a}_1 + M_2\mathbf{a}_2 + M_3\mathbf{a}_3 \quad [3.42]$$

where symbols keep the meaning already introduced. Since $\mathbf{R}\dot{\mathbf{e}} = \dot{\varepsilon}\mathbf{a}_1 + \dot{\gamma}_2\mathbf{a}_2 + \dot{\gamma}_3\mathbf{a}_3$ and $\mathbf{R}\dot{\boldsymbol{\chi}} = \dot{\chi}_1\mathbf{a}_1 + \dot{\chi}_2\mathbf{a}_2 + \dot{\chi}_3\mathbf{a}_3$, it is also:

$$\mathcal{P}_{int} = \int_{\mathcal{S}} (N\dot{\varepsilon} + T_2\dot{\gamma}_2 + T_3\dot{\gamma}_3 + M_1\dot{\chi}_1 + M_2\dot{\chi}_2 + M_3\dot{\chi}_3)\, ds \quad [3.43]$$

The VPP [2.100] states that $\mathcal{P}_{ext} = \mathcal{P}_{int}$, $\forall\,(\mathbf{v},\boldsymbol{\omega})$; from this, by performing the same steps as for the straight beam, the same vector form of balance equation [2.107] is derived:

$$\mathbf{t}' + \mathbf{p} = m\ddot{\mathbf{u}}$$
$$\mathbf{m}' + \mathbf{x}'\times\mathbf{t} + \mathbf{c} = \mathbf{I}_G\dot{\boldsymbol{\omega}} \quad [3.44]$$

in which inertia forces have been introduced via the d'Alembert principle. They express the linear and angular momentum balance equations, respectively. The VPP also furnishes the natural boundary conditions, via the alternatives [2.104]:

$$(\mathbf{P}_H \pm \mathbf{t}_H)\cdot\mathbf{v}_H = 0$$
$$(\mathbf{C}_H \pm \mathbf{m}_H)\cdot\boldsymbol{\omega}_H, \qquad H = A, B \quad [3.45]$$

Scalar balance equations in the current and reference bases

When the balance equations are projected onto the current basis $\mathcal{B}$, the matrix equations [2.136] and [2.138] still hold, in which, however, the curvature $\boldsymbol{K}$ must be replaced by $\bar{\boldsymbol{K}} + \boldsymbol{X}$, in order for strain measures to appear. Consequently:

$$\boldsymbol{t}' + \left(\bar{\boldsymbol{K}} + \boldsymbol{X}\right)\boldsymbol{t} + \boldsymbol{R}^T\bar{\boldsymbol{p}} = m\boldsymbol{R}^T\ddot{\mathbf{u}}$$
$$\boldsymbol{m}' + \left(\bar{\boldsymbol{K}} + \boldsymbol{X}\right)\boldsymbol{m} + \boldsymbol{\Lambda}\boldsymbol{t} + \boldsymbol{R}^T\bar{\boldsymbol{c}} = \boldsymbol{I}_G\dot{\boldsymbol{\omega}} \quad [3.46]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where $\boldsymbol{\Lambda}$ is defined in equation [2.85]. Concerning the mechanical boundary conditions, no changes are requested, so that they read:

$$\mp \boldsymbol{t}_H = \boldsymbol{R}_H^T \bar{\boldsymbol{P}}_H, \qquad \mp \boldsymbol{m}_H = \boldsymbol{R}_H^T \bar{\boldsymbol{C}}_H \tag{3.47}$$

When projections are made onto the reference basis $\bar{\mathcal{B}}$ the field equations read:

$$\begin{aligned} &\boldsymbol{R}\left[\boldsymbol{t}' + \left(\bar{\boldsymbol{K}} + \boldsymbol{X}\right)\boldsymbol{t}\right] + \bar{\boldsymbol{p}} = m\ddot{\boldsymbol{u}} \\ &\boldsymbol{R}\left[\boldsymbol{m}' + \left(\bar{\boldsymbol{K}} + \boldsymbol{X}\right)\boldsymbol{m}\right] + \boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{t} + \bar{\boldsymbol{c}} = \boldsymbol{R}\boldsymbol{I}_G\dot{\boldsymbol{\omega}} \end{aligned} \tag{3.48}$$

and the boundary conditions:

$$\mp \boldsymbol{R}_H \boldsymbol{t}_H = \bar{\boldsymbol{P}}_H, \qquad \mp \boldsymbol{R}_H \boldsymbol{m}_H = \bar{\boldsymbol{C}}_H \tag{3.49}$$

3.2.3 The elastic law

If the material is hyperelastic and the simplest uncoupled linear law is adopted, then:

$$\begin{aligned} &N = \overset{\circ}{N} + EA\varepsilon, \qquad T_2 = \overset{\circ}{T}_2 + GA_2\gamma_2, \qquad T_3 = \overset{\circ}{T}_3 + GA_3\gamma_3 \\ &M_1 = \overset{\circ}{M}_1 + GJ_1\chi_1, \quad M_2 = \overset{\circ}{M}_2 + EJ_2\chi_2, \quad M_3 = \overset{\circ}{M}_3 + EJ_3\chi_3 \end{aligned} \tag{3.50}$$

where EA, GA_j, GJ_1, EJ_j are elastic coefficients. These equations are identical to equation [2.164], valid for the straight beam, but with the changes of curvature χ_i replacing κ_i.

3.2.4 The Fundamental Problem

The Fundamental Problem for the curved beam is governed by eighteen equations in the eighteen unknowns $\boldsymbol{u}$,$\boldsymbol{\theta}$ $\boldsymbol{e}$, $\boldsymbol{\chi}$, $\boldsymbol{t}$, $\boldsymbol{m}$, namely:

– the six strain-displacement relationships [3.31] and [3.34];

– the six balance equations [3.46], or [3.48];

– the six constitutive laws [3.50], if the material is linearly elastic.

These equations must be joined to the geometric boundary conditions [3.19] and/or the mechanical boundary conditions [3.47], or [3.49].

The equations of motion for unprestressed beams

When the governing equations are combined among them, the equations of motion are derived. In the following we will limit ourselves to the unprestressed case, to avoid very cumbersome expressions, and we will use the scalar representation of the balance equations and relevant boundary conditions in the current configuration, where equations assume a simpler form.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The balance of forces [3.46a] reads:

$$\begin{aligned}
&\{EA[((1+u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)\cos\theta_1\cos\theta_3 + (u_2' + \bar{\kappa}_3 u_1 \\
&\quad - \bar{\kappa}_1 u_3)\cos\theta_2\sin\theta_3 - (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2)\sin\theta_2 - 1]\}' \\
&\quad - GA_2[\theta_3'\cos\theta_1\cos\theta_2 - \theta_2'\sin\theta_1 + \bar{\kappa}_1(\cos\theta_1\sin\theta_2\cos\theta_3 \\
&\quad + \sin\theta_1\sin\theta_3) + \bar{\kappa}_2(\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3) \\
&\quad + \bar{\kappa}_3\cos\theta_1\cos\theta_2][((1+u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\quad - \cos\theta_1\sin\theta_3) + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)(\cos\theta_1\cos\theta_3 \\
&\quad + \sin\theta_1\sin\theta_2\sin\theta_3) + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2)\sin\theta_1\cos\theta_2] \\
&\quad + GA_3[\theta_2'\cos\theta_1 + \theta_3'\sin\theta_1\cos\theta_2 + \bar{\kappa}_1(\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\quad - \cos\theta_1\sin\theta_3) + \bar{\kappa}_2(\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3) \\
&\quad + \bar{\kappa}_3\sin\theta_1\cos\theta_2][((1+u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\cos\theta_1\sin\theta_2\cos\theta_3 \\
&\quad + \sin\theta_1\sin\theta_3) + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)(\cos\theta_1\sin\theta_2\sin\theta_3 \\
&\quad - \sin\theta_1\cos\theta_3) + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2)\cos\theta_1\cos\theta_2] \\
&\quad - (m\ddot{u}_1 - \bar{p}_1)\cos\theta_1\cos\theta_3 - (m\ddot{u}_2 - \bar{p}_2)\cos\theta_2\sin\theta_3 \\
&\quad + (m\ddot{u}_3 - \bar{p}_3)\sin\theta_2 = 0
\end{aligned} \qquad [3.51]$$

$$\begin{aligned}
&\{GA_2[((1+u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3) \\
&\quad + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)(\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3) \\
&\quad + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2)\sin\theta_1\cos\theta_2]\}' + EA[\theta_3'\cos\theta_1\cos\theta_2 \\
&\quad - \theta_2'\sin\theta_1 + \bar{\kappa}_1(\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3) + \bar{\kappa}_2(\cos\theta_1\sin\theta_2\sin\theta_3 \\
&\quad - \sin\theta_1\cos\theta_3) + \bar{\kappa}_3\cos\theta_1\cos\theta_2][((1+u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)\cos\theta_1\cos\theta_3 \\
&\quad + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)\cos\theta_2\sin\theta_3 - (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2)\sin\theta_2 - 1] \\
&\quad - GA_3[\theta_1' - \theta_3'\sin\theta_2 + \bar{\kappa}_1\cos\theta_1\cos\theta_3 + \bar{\kappa}_2\cos\theta_2\sin\theta_3 - \bar{\kappa}_3\sin\theta_2][((1+u_1') \\
&\quad - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3) + (u_2' + \bar{\kappa}_3 u_1 \\
&\quad - \bar{\kappa}_1 u_3)(\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3) + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2)\cos\theta_1\cos\theta_2] \\
&\quad - (m\ddot{u}_1 - \bar{p}_1)(\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3) - (m\ddot{u}_2 - \bar{p}_2)(\cos\theta_1\cos\theta_3 \\
&\quad + \sin\theta_1\sin\theta_2\sin\theta_3) - (m\ddot{u}_3 - \bar{p}_3)\sin\theta_1\cos\theta_2 = 0
\end{aligned}$$

[3.52]

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$
\begin{aligned}
&\{GA_3[((1+u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3) \\
&\quad + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3) + (u_3' - \bar{\kappa}_2 u_1 \\
&\quad + \bar{\kappa}_1 u_2) \cos\theta_1 \cos\theta_2]\}' - EA[\theta_2' \cos\theta_1 \\
&\quad + \theta_3' \sin\theta_1 \cos\theta_2 + \bar{\kappa}_1(\sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad - \cos\theta_1 \sin\theta_3) + \bar{\kappa}_2(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3) \\
&\quad + \bar{\kappa}_3 \sin\theta_1 \cos\theta_2][((1+u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3) \cos\theta_1 \cos\theta_3 \\
&\quad + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3) \cos\theta_2 \sin\theta_3 - (u_3' - \bar{\kappa}_2 u_1 \\
&\quad + \bar{\kappa}_1 u_2) \sin\theta_2 - 1] + GA_2[\theta_1' - \theta_3' \sin\theta_2 + \bar{\kappa}_1 \cos\theta_1 \cos\theta_3 + \bar{\kappa}_2 \cos\theta_2 \sin\theta_3 \\
&\quad - \bar{\kappa}_3 \sin\theta_2 - \bar{\kappa}_1][((1+u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad - \cos\theta_1 \sin\theta_3) + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)(\cos\theta_1 \cos\theta_3 \\
&\quad + \sin\theta_1 \sin\theta_2 \sin\theta_3) + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2) \sin\theta_1 \cos\theta_2] \\
&\quad - (m\ddot{u}_1 - \bar{p}_1)(\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3) \\
&\quad - (m\ddot{u}_2 - \bar{p}_2)(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3) \\
&\quad - (m\ddot{u}_3 - \bar{p}_3) \cos\theta_1 \cos\theta_2 = 0
\end{aligned}
\qquad [3.53]
$$

Concerning the balance of moments, equation [3.46b], we have:

$$
\begin{aligned}
&\{GJ_1[\theta_1' - \theta_3' \sin\theta_2 + \bar{\kappa}_1 \cos\theta_1 \cos\theta_3 + \bar{\kappa}_2 \cos\theta_2 \sin\theta_3 - \bar{\kappa}_3 \sin\theta_2 - \bar{\kappa}_1]\}' \\
&\quad - EJ_2[\theta_2' \cos\theta_1 + \theta_3' \sin\theta_1 \cos\theta_2 + \bar{\kappa}_1(\sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad - \cos\theta_1 \sin\theta_3) + \bar{\kappa}_2(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3) \\
&\quad + \bar{\kappa}_3 \sin\theta_1 \cos\theta_2 - \bar{\kappa}_2][\theta_3' \cos\theta_1 \cos\theta_2 - \theta_2' \sin\theta_1 + \bar{\kappa}_1(\cos\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad + \sin\theta_1 \sin\theta_3) + \bar{\kappa}_2(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3) + \bar{\kappa}_3 \cos\theta_1 \cos\theta_2] \\
&\quad + EJ_3[\theta_3' \cos\theta_1 \cos\theta_2 - \theta_2' \sin\theta_1 + \bar{\kappa}_1(\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3) \\
&\quad + \bar{\kappa}_2(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3) + \bar{\kappa}_3 \cos\theta_1 \cos\theta_2 \\
&\quad - \bar{\kappa}_3][\theta_2' \cos\theta_1 + \theta_3' \sin\theta_1 \cos\theta_2 + \bar{\kappa}_1(\sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad - \cos\theta_1 \sin\theta_3) + \bar{\kappa}_2(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3) \\
&\quad + \bar{\kappa}_3 \sin\theta_1 \cos\theta_2] + (GA_3 - GA_2)[((1+u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad - \cos\theta_1 \sin\theta_3) + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)(\cos\theta_1 \cos\theta_3 \\
&\quad + \sin\theta_1 \sin\theta_2 \sin\theta_3) + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2) \sin\theta_1 \cos\theta_2][((1+u_1') \\
&\quad - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3) \\
&\quad + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3) \\
&\quad + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2) \cos\theta_1 \cos\theta_2] - I_1(\dot{\theta}_1 - \theta_3' \sin\theta_2)^{\cdot} \\
&\quad + \bar{c}_1 \cos\theta_1 \cos\theta_3 + \bar{c}_2 \cos\theta_2 \sin\theta_3 + \bar{c}_3 \sin\theta_2 = 0
\end{aligned}
\qquad [3.54]
$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$
\begin{aligned}
&\{EJ_2[\theta_2' \cos\theta_1 + \theta_3' \sin\theta_1 \cos\theta_2 + \bar{\kappa}_1(\sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3)\\
&\quad + \bar{\kappa}_2(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3) + \bar{\kappa}_3 \sin\theta_1 \cos\theta_2 - \bar{\kappa}_2]\}'\\
&\quad + GJ_1[\theta_1' - \theta_3' \sin\theta_2 + \bar{\kappa}_1 \cos\theta_1 \cos\theta_3 + \bar{\kappa}_2 \cos\theta_2 \sin\theta_3\\
&\quad - \bar{\kappa}_3 \sin\theta_2 - \bar{\kappa}_1][\theta_3' \cos\theta_1 \cos\theta_2 - \theta_2' \sin\theta_1 + \bar{\kappa}_1(\cos\theta_1 \sin\theta_2 \cos\theta_3\\
&\quad + \sin\theta_1 \sin\theta_3) + \bar{\kappa}_2(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3)\\
&\quad + \bar{\kappa}_3 \cos\theta_1 \cos\theta_2] - GJ_1[\theta_1' - \theta_3' \sin\theta_2 + \bar{\kappa}_1 \cos\theta_1 \cos\theta_3\\
&\quad + \bar{\kappa}_2 \cos\theta_2 \sin\theta_3 - \bar{\kappa}_3 \sin\theta_2 - \bar{\kappa}_1][\theta_3' \cos\theta_1 \cos\theta_2 - \theta_2' \sin\theta_1\\
&\quad + \bar{\kappa}_1(\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3) + \bar{\kappa}_2(\cos\theta_1 \sin\theta_2 \sin\theta_3\\
&\quad - \sin\theta_1 \cos\theta_3) + \bar{\kappa}_3 \cos\theta_1 \cos\theta_2] + (EA - GA_3)[((1 + u_1') - \bar{\kappa}_3 u_2\\
&\quad + \bar{\kappa}_2 u_3) \cos\theta_1 \cos\theta_3 + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3) \cos\theta_2 \sin\theta_3 - (u_3' - \bar{\kappa}_2 u_1\\
&\quad + \bar{\kappa}_1 u_2) \sin\theta_2 - 1][((1 + u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3)\\
&\quad + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3)\\
&\quad + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2) \cos\theta_1 \cos\theta_2] - GA_3[((1 + u_1') - \bar{\kappa}_3 u_2\\
&\quad + \bar{\kappa}_2 u_3)(\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3) + (u_2' + \bar{\kappa}_3 u_1\\
&\quad - \bar{\kappa}_1 u_3)(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3) + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2) \cos\theta_1 \cos\theta_2]\\
&\quad - I_2(\dot{\theta}_2 \cos\theta_1 + \dot{\theta}_3 \sin\theta_1 \cos\theta_2)^{\cdot}\\
&\quad + \bar{c}_1(\sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3) + \bar{c}_2(\cos\theta_1 \cos\theta_3\\
&\quad + \sin\theta_1 \sin\theta_2 \sin\theta_3) + \bar{c}_3 \sin\theta_1 \cos\theta_2 = 0
\end{aligned}
\qquad [3.55]
$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$
\begin{aligned}
&\{EJ_3[\theta_3' \cos\theta_1 \cos\theta_2 - \theta_2' \sin\theta_1 + \bar{\kappa}_1(\cos\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad + \sin\theta_1 \sin\theta_3) + \bar{\kappa}_2(\cos\theta_1 \sin\theta_2 \sin\theta_3 - \sin\theta_1 \cos\theta_3) \\
&\quad + \bar{\kappa}_3 \cos\theta_1 \cos\theta_2 - \bar{\kappa}_3]\}' - GJ_1[\theta_1' - \theta_3' \sin\theta_2 + \bar{\kappa}_1 \cos\theta_1 \cos\theta_3 \\
&\quad + \bar{\kappa}_2 \cos\theta_2 \sin\theta_3 - \bar{\kappa}_3 \sin\theta_2 - \bar{\kappa}_1][\theta_2' \cos\theta_1 + \theta_3' \sin\theta_1 \cos\theta_2 \\
&\quad + \bar{\kappa}_1(\sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3) + \bar{\kappa}_2(\cos\theta_1 \cos\theta_3 \\
&\quad + \sin\theta_1 \sin\theta_2 \sin\theta_3) + \bar{\kappa}_3 \sin\theta_1 \cos\theta_2] - EI_2[\theta_2' \cos\theta_1 + \theta_3' \sin\theta_1 \cos\theta_2 \\
&\quad + \bar{\kappa}_1(\sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3) + \bar{\kappa}_2(\cos\theta_1 \cos\theta_3 \\
&\quad + \sin\theta_1 \sin\theta_2 \sin\theta_3) + \bar{\kappa}_3 \sin\theta_1 \cos\theta_2 - \bar{\kappa}_2][\theta_1' - \theta_3' \sin\theta_2 \\
&\quad + \bar{\kappa}_1 \cos\theta_1 \cos\theta_3 + \bar{\kappa}_2 \cos\theta_2 \sin\theta_3 - \bar{\kappa}_3 \sin\theta_2] \\
&\quad + (GA_2 - EA)[((1 + u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3) \cos\theta_1 \cos\theta_3 + (u_2' + \bar{\kappa}_3 u_1 \\
&\quad - \bar{\kappa}_1 u_3) \cos\theta_2 \sin\theta_3 - (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2) \sin\theta_2 - 1][((1 + u_1') \\
&\quad - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\sin\theta_1 \sin\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_3) \\
&\quad + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3) \\
&\quad + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2) \sin\theta_1 \cos\theta_2] + GA_2[((1 + u_1') - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3)(\sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\quad - \cos\theta_1 \sin\theta_3) + (u_2' + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3)(\cos\theta_1 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3) \\
&\quad + (u_3' - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2) \sin\theta_1 \cos\theta_2] - I_3(\dot{\theta}_3 \cos\theta_1 \cos\theta_2 \\
&\quad - \dot{\theta}_2 \sin\theta_1)^{\cdot} + \bar{c}_1(\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3) + \bar{c}_2(\cos\theta_1 \sin\theta_2 \sin\theta_3 \\
&\quad + \sin\theta_1 \cos\theta_3) - \bar{c}_3 \cos\theta_1 \cos\theta_2
\end{aligned}
$$

[3.56]

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The mechanical boundary conditions [3.47] become (forces):

$$
\begin{aligned}
&\{EA[((1+u_1')-\bar\kappa_3u_2+\bar\kappa_2u_3)\cos\theta_1\cos\theta_3+(u_2'+\bar\kappa_3u_1\\
&\quad-\bar\kappa_1u_3)\cos\theta_2\sin\theta_3-(u_3'-\bar\kappa_2u_1+\bar\kappa_1u_2)\sin\theta_2-1]\}_H\\
&\quad=\bar P_{1H}(\cos\theta_1\cos\theta_3)_H+\bar P_{2H}(\cos\theta_2\sin\theta_3)_H-\bar P_{3H}(\sin\theta_2)_H\\
&\{GA_2[((1+u_1')-\bar\kappa_3u_2+\bar\kappa_2u_3)(\sin\theta_1\sin\theta_2\cos\theta_3-\cos\theta_1\sin\theta_3)\\
&\quad+(u_2'+\bar\kappa_3u_1-\bar\kappa_1u_3)(\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3)+(u_3'-\bar\kappa_2u_1\\
&\quad+\bar\kappa_1u_2)\sin\theta_1\cos\theta_2]\}_H=\bar P_{1H}(\sin\theta_1\sin\theta_2\cos\theta_3-\cos\theta_1\sin\theta_3)_H\\
&\quad+\bar P_{2H}(\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3)_H+\bar P_{3H}(\sin\theta_1\cos\theta_2)_H\\
&\{GA_3[((1+u_1')-\bar\kappa_3u_2+\bar\kappa_2u_3)(\cos\theta_1\sin\theta_2\cos\theta_3+\sin\theta_1\sin\theta_3)\\
&\quad+(u_2'+\bar\kappa_3u_1-\bar\kappa_1u_3)(\cos\theta_1\sin\theta_2\sin\theta_3-\sin\theta_1\cos\theta_3)+(u_3'-\bar\kappa_2u_1\\
&\quad+\bar\kappa_1u_2)\cos\theta_1\cos\theta_2]\}_H=\bar P_{1H}(\cos\theta_1\sin\theta_2\cos\theta_3+\sin\theta_1\sin\theta_3)_H\\
&\quad+\bar P_{2H}(\cos\theta_1\sin\theta_2\sin\theta_3-\sin\theta_1\cos\theta_3)_H+\bar P_{3H}(\cos\theta_1\cos\theta_2)_H
\end{aligned}
\qquad [3.57]
$$

and (moments):

$$
\begin{aligned}
&\{GJ_1[\theta_1'-\theta_3'\sin\theta_2+\bar\kappa_1\cos\theta_1\cos\theta_3+\bar\kappa_2\cos\theta_2\sin\theta_3-\bar\kappa_3\sin\theta_2-\bar\kappa_1]\}_H\\
&\quad=\bar C_{1H}(\cos\theta_1\cos\theta_3)_H+\bar C_{2H}(\cos\theta_2\sin\theta_3)_H\\
&\quad-\bar C_{3H}(\sin\theta_2)_H\\
&\{EJ_2[\theta_2'\cos\theta_1+\theta_3'\sin\theta_1\cos\theta_2+\bar\kappa_1(\sin\theta_1\sin\theta_2\cos\theta_3-\cos\theta_1\sin\theta_3)\\
&\quad+\bar\kappa_2(\cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3)+\bar\kappa_3\sin\theta_1\cos\theta_2-\bar\kappa_2]\}_H\\
&\quad=\bar C_{1H}(\sin\theta_1\sin\theta_2\cos\theta_3-\cos\theta_1\sin\theta_3)_H+\bar C_{2H}(\cos\theta_1\cos\theta_3\\
&\quad+\sin\theta_1\sin\theta_2\sin\theta_3)_H+\bar C_{3H}(\sin\theta_1\cos\theta_2)_H\\
&\{EJ_3[\theta_3'\cos\theta_1\cos\theta_2-\theta_2'\sin\theta_1+\bar\kappa_1(\cos\theta_1\sin\theta_2\cos\theta_3+\sin\theta_1\sin\theta_3)\\
&\quad+\bar\kappa_2(\cos\theta_1\sin\theta_2\sin\theta_3-\sin\theta_1\cos\theta_3)+\bar\kappa_3\cos\theta_1\cos\theta_2-\bar\kappa_3]\}_H\\
&\quad=\bar C_{1H}(\cos\theta_1\sin\theta_2\cos\theta_3+\sin\theta_1\sin\theta_3)_H+\bar C_{2H}(\cos\theta_1\sin\theta_2\sin\theta_3\\
&\quad-\sin\theta_1\cos\theta_3)_H+\bar C_{3H}(\cos\theta_1\cos\theta_2)_H
\end{aligned}
\qquad [3.58]
$$

while the geometric boundary conditions are expressed by equation [3.19].

Linearized equations for elastic prestressed beams

If the beam is prestressed, and we confine ourselves to the linearized theory, the governing equations reduce to:

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

1) Linearized strains and changes of curvature:

$$\begin{aligned}
\varepsilon &= u_1' - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3 \\
\gamma_2 &= u_2' - \theta_3 + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3 \\
\gamma_3 &= u_3' + \theta_2 - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2 \\
\chi_1 &= \theta_1' - \bar{\kappa}_3 \theta_2 + \bar{\kappa}_2 \theta_3 \\
\chi_2 &= \theta_2' + \bar{\kappa}_3 \theta_1 - \bar{\kappa}_1 \theta_3 \\
\chi_3 &= \theta_3' - \bar{\kappa}_2 \theta_1 + \bar{\kappa}_1 \theta_2
\end{aligned} \tag{3.59}$$

2) Linearized balance equations in incremental form (forces):

$$\begin{aligned}
&\tilde{N}' - \bar{\kappa}_3 \tilde{T}_2 + \bar{\kappa}_2 \tilde{T}_3 - [\theta_3' - \bar{\kappa}_2 \theta_1 + \bar{\kappa}_1 \theta_2] \mathring{T}_2 \\
&\quad + [\theta_2' + \bar{\kappa}_3 \theta_1 - \bar{\kappa}_1 \theta_3] \mathring{T}_3 - m \ddot{u}_1 + \tilde{p}_1 = 0 \\
&\tilde{T}_2' + \bar{\kappa}_3 \tilde{N} - \bar{\kappa}_1 \tilde{T}_3 + [\theta_3' - \bar{\kappa}_2 \theta_1 + \bar{\kappa}_1 \theta_2] \mathring{N} \\
&\quad - [\theta_1' - \bar{\kappa}_3 \theta_2 + \bar{\kappa}_2 \theta_3] \mathring{T}_3 - m \ddot{u}_2 + \tilde{p}_2 = 0 \\
&\tilde{T}_3' - \bar{\kappa}_2 \tilde{N} + \bar{\kappa}_1 \tilde{T}_2 - [\theta_2' + \bar{\kappa}_3 \theta_1 - \bar{\kappa}_1 \theta_3] \mathring{N} \\
&\quad + [\theta_1' - \bar{\kappa}_3 \theta_2 + \bar{\kappa}_2 \theta_3] \mathring{T}_2 - m \ddot{u}_3 + \tilde{p}_3 = 0
\end{aligned} \tag{3.60}$$

and (moments):

$$\begin{aligned}
&\tilde{M}_1' - \bar{\kappa}_3 \tilde{M}_2 + \bar{\kappa}_2 \tilde{M}_3 - [\theta_3' - \bar{\kappa}_2 \theta_1 + \bar{\kappa}_1 \theta_2] \mathring{M}_2 \\
&\quad + [\theta_2' + \bar{\kappa}_3 \theta_1 - \bar{\kappa}_1 \theta_3] \mathring{M}_3 - [u_3' + \theta_2 - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2] \mathring{T}_2 \\
&\quad + [u_2' - \theta_3 + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3] \mathring{T}_3 - I_1 \ddot{\theta}_1 + \tilde{c}_1 = 0 \\
&\tilde{M}_2' + \bar{\kappa}_3 \tilde{M}_1 - \bar{\kappa}_1 \tilde{M}_3 + [\theta_3' - \bar{\kappa}_2 \theta_1 + \bar{\kappa}_1 \theta_2] \mathring{M}_1 \\
&\quad - [\theta_1' - \bar{\kappa}_3 \theta_2 + \bar{\kappa}_2 \theta_3] \mathring{M}_3 - \tilde{T}_3 + [u_3' + \theta_2 - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2] \mathring{N} \\
&\quad - [u_1' - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3] \mathring{T}_3 - I_2 \ddot{\theta}_2 + \tilde{c}_2 = 0 \\
&\tilde{M}_3' - \bar{\kappa}_2 \tilde{M}_1 + \bar{\kappa}_1 \tilde{M}_2 - [\theta_2' + \bar{\kappa}_3 \theta_1 - \bar{\kappa}_1 \theta_3] \mathring{M}_1 \\
&\quad + [\theta_1' - \bar{\kappa}_3 \theta_2 + \bar{\kappa}_2 \theta_3] \mathring{M}_2 + \tilde{T}_2 - [u_2' - \theta_3 + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3] \mathring{N} \\
&\quad + [u_1' - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3] \mathring{T}_2 - I_3 \ddot{\theta}_3 + \tilde{c}_3 = 0
\end{aligned} \tag{3.61}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with the mechanical boundary conditions:

$$\begin{aligned}
&\{\tilde{N} - \theta_3 \mathring{T}_2 + \theta_2 \mathring{T}_3\}_H = \tilde{P}_{1H} \\
&\{\tilde{T}_2 + \theta_3 \mathring{N} - \theta_1 \mathring{T}_3\}_H = \tilde{P}_{2H} \\
&\{\tilde{T}_3 - \theta_2 \mathring{N} + \theta_1 \mathring{T}_2\}_H = \tilde{P}_{3H} \\
&\{\tilde{M}_1 - \theta_3 \mathring{M}_2 + \theta_2 \mathring{M}_3\}_H = \tilde{C}_{1H} \\
&\{\tilde{M}_2 + \theta_3 \mathring{M}_1 - \theta_1 \mathring{M}_3\}_H = \tilde{C}_{2H} \\
&\{\tilde{M}_3 - \theta_2 \mathring{M}_1 + \theta_1 \mathring{M}_2\}_H = \tilde{C}_{3H}
\end{aligned} \qquad [3.62]$$

Here, as usual, the symbol over-ring ($\mathring{\ }$) denotes a prestress or a preload and a tilde ($\tilde{\ }$) denotes a small increment; moreover, equilibrium at reference configuration has been taken into account.

3) Linear elastic laws for the incremental stresses:

$$\begin{aligned}
&\tilde{N} = EA\varepsilon, \qquad \tilde{T}_2 = GA_2\gamma_2, \quad \tilde{T}_3 = GA_3\gamma_3 \\
&\tilde{M}_1 = GJ_1\chi_1, \quad \tilde{M}_2 = EJ_2\chi_2, \quad \tilde{M}_3 = EJ_3\chi_3
\end{aligned} \qquad [3.63]$$

From the previous equations, the following equations of motion are derived for the forces:

$$\begin{aligned}
&\{EA[u_1' - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3]\}' - \bar{\kappa}_3 GA_2[u_2' - \theta_3 + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3] \\
&\qquad + \bar{\kappa}_2 GA_3[u_3' + \theta_2 - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2] - [\theta_3' - \bar{\kappa}_2\theta_1 + \bar{\kappa}_1\theta_2]\mathring{T}_2 \\
&\qquad + [\theta_2' + \bar{\kappa}_3\theta_1 - \bar{\kappa}_1\theta_3]\mathring{T}_3 - m\ddot{u}_1 + \tilde{p}_1 = 0 \\
&\{GA_2[u_2' - \theta_3 + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3]\}' + \bar{\kappa}_3 EA[u_1' - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3] \\
&\qquad - \bar{\kappa}_1 GA_3[u_3' + \theta_2 - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2] + [\theta_3' - \bar{\kappa}_2\theta_1 + \bar{\kappa}_1\theta_2]\mathring{N} \\
&\qquad - [\theta_1' - \bar{\kappa}_3\theta_2 + \bar{\kappa}_2\theta_3]\mathring{T}_3 - m\ddot{u}_2 + \tilde{p}_2 = 0 \\
&\{GA_3[u_3' + \theta_2 - \bar{\kappa}_2 u_1 + \bar{\kappa}_1 u_2]\}' - \bar{\kappa}_2 EA[u_1' - \bar{\kappa}_3 u_2 + \bar{\kappa}_2 u_3] \\
&\qquad + \bar{\kappa}_1 GA_2[u_2' - \theta_3 + \bar{\kappa}_3 u_1 - \bar{\kappa}_1 u_3] - [\theta_2' + \bar{\kappa}_3\theta_1 - \bar{\kappa}_1\theta_3]\mathring{N} \\
&\qquad + [\theta_1' - \bar{\kappa}_3\theta_2 + \bar{\kappa}_2\theta_3]\mathring{T}_2 - m\ddot{u}_3 + \tilde{p}_3 = 0
\end{aligned} \qquad [3.64]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and the following equations of motion are derived for the moments:

$$
\begin{aligned}
&\{GJ_1[\theta_1' - \bar\kappa_3\theta_2 + \bar\kappa_2\theta_3]\}' - \bar\kappa_3 EJ_2[\theta_2' + \bar\kappa_3\theta_1 - \bar\kappa_1\theta_3] \\
&\quad + \bar\kappa_2 EJ_3[\theta_3' - \bar\kappa_2\theta_1 + \bar\kappa_1\theta_2] - [\theta_3' - \bar\kappa_2\theta_1 + \bar\kappa_1\theta_2]\mathring{M}_2 \\
&\quad + [\theta_2' + \bar\kappa_3\theta_1 - \bar\kappa_1\theta_3]\mathring{M}_3 - [u_3' + \theta_2 - \bar\kappa_2 u_1 + \bar\kappa_1 u_2]\mathring{T}_2 \\
&\quad + [u_2' - \theta_3 + \bar\kappa_3 u_1 - \bar\kappa_1 u_3]\mathring{T}_3 - I_1\ddot\theta_1 + \tilde c_1 = 0 \\
&\{EJ_2[\theta_2' + \bar\kappa_3\theta_1 - \bar\kappa_1\theta_3]\}' + \bar\kappa_3 GJ_1[\theta_1' - \bar\kappa_3\theta_2 + \bar\kappa_2\theta_3] \\
&\quad - \bar\kappa_1 EJ_3[\theta_3' - \bar\kappa_2\theta_1 + \bar\kappa_1\theta_2] + [\theta_3' - \bar\kappa_2\theta_1 + \bar\kappa_1\theta_2]\mathring{M}_1 \\
&\quad - [\theta_1' - \bar\kappa_3\theta_2 + \bar\kappa_2\theta_3]\mathring{M}_3 - GA_3[u_3' + \theta_2 - \bar\kappa_2 u_1 + \bar\kappa_1 u_2] \\
&\quad + [u_3' + \theta_2 - \bar\kappa_2 u_1 + \bar\kappa_1 u_2]\mathring{N} - [u_1' - \bar\kappa_3 u_2 + \bar\kappa_2 u_3]\mathring{T}_3 - I_2\ddot\theta_2 + \tilde c_2 = 0 \\
&\{EJ_3[\theta_3' - \bar\kappa_2\theta_1 + \bar\kappa_1\theta_2]\}' - \bar\kappa_2 GJ_1[\theta_1' - \bar\kappa_3\theta_2 + \bar\kappa_2\theta_3] \\
&\quad + \bar\kappa_1 EJ_2[\theta_2' + \bar\kappa_3\theta_1 - \bar\kappa_1\theta_3] - [\theta_2' + \bar\kappa_3\theta_1 - \bar\kappa_1\theta_3]\mathring{M}_1 \\
&\quad + [\theta_1' - \bar\kappa_3\theta_2 + \bar\kappa_2\theta_3]\mathring{M}_2 + GA_2[u_2' - \theta_3 + \bar\kappa_3 u_1 - \bar\kappa_1 u_3] \\
&\quad - [u_2' - \theta_3 + \bar\kappa_3 u_1 - \bar\kappa_1 u_3]\mathring{N} + [u_1' - \bar\kappa_3 u_2 + \bar\kappa_2 u_3]\mathring{T}_2 - I_3\ddot\theta_3 + \tilde c_3 = 0
\end{aligned}
\tag{3.65}
$$

The boundary conditions are[7]:

$$
\begin{aligned}
&[EA(u_1' - \bar\kappa_3 u_2 + \bar\kappa_2 u_3) - \theta_3\mathring{T}_2 + \theta_2\mathring{T}_3]_H = \tilde P_{1H} \\
&[GA_2(u_2' - \theta_3 + \bar\kappa_3 u_1 - \bar\kappa_1 u_3) + \theta_3\mathring{N} - \theta_1\mathring{T}_3]_H = \tilde P_{2H} \\
&[GA_3(u_3' + \theta_2 - \bar\kappa_2 u_1 + \bar\kappa_1 u_2) - \theta_2\mathring{N} + \theta_1\mathring{T}_2]_H = \tilde P_{3H} \\
&[GJ_1(\theta_1' - \bar\kappa_3\theta_2 + \bar\kappa_2\theta_3) - \theta_3\mathring{M}_2 + \theta_2\mathring{M}_3]_H = \tilde C_{1H} \\
&[EJ_2(\theta_2' + \bar\kappa_3\theta_1 - \bar\kappa_1\theta_3) + \theta_3\mathring{M}_1 - \theta_1\mathring{M}_3]_H = \tilde C_{2H} \\
&[EJ_3(\theta_3' - \bar\kappa_2\theta_1 + \bar\kappa_1\theta_2) - \theta_2\mathring{M}_1 + \theta_1\mathring{M}_2]_H = \tilde C_{3H}
\end{aligned}
\tag{3.66}
$$

3.3 The planar curved beam

Let us consider a curved beam, whose centerline, in the reference configuration, lies on a curve $\mathcal{S}$ of the plane π that is spanned by the $(\mathbf{i}_1, \mathbf{i}_2)$ unit vectors. Let us

7. These equations, in matrix form, become $\boldsymbol{L}\boldsymbol{u} + \boldsymbol{G}\boldsymbol{u} = \tilde{\boldsymbol{p}}$, $\boldsymbol{\mathcal{L}}_H\boldsymbol{u} + \boldsymbol{\mathcal{G}}_H\boldsymbol{u} = \tilde{\boldsymbol{P}}$ (equation [1.72]).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

assume that π is a principal plane of inertia for any cross-sections. Moreover, let the constraints at the boundary allow the beam to assume configurations still contained in π. Such a curved beam is said to be *planar* or an *arch*.

The initial curvature

Since $\mathcal{S}$ is planar, the position vector in the reference configuration is $\bar{\mathbf{x}} = \bar{x}_1(s)\mathbf{i}_1 + \bar{x}_2(s)\mathbf{i}_2$; moreover, since the osculating plane coincides with π for any s, then the tangent and normal vector to the curve also belong to π, while the binormal vector is orthogonal to it:

$$\mathbf{a}_t = \bar{\mathbf{x}}', \qquad \mathbf{a}_n = \frac{1}{k}\bar{\mathbf{x}}'', \qquad \mathbf{a}_b = \mathbf{i}_3 \qquad [3.67]$$

with $k = \|\bar{\mathbf{x}}''\|$ and $\tau = 0$. As π is a principal plane of inertia, the deviation angle $\delta(s)$ vanishes everywhere. Therefore, from equation [3.4], *the triad of directors* $\bar{\mathcal{B}} = (\bar{\mathbf{a}}_1(s), \bar{\mathbf{a}}_2(s), \bar{\mathbf{a}}_3)$ *attached to the cross-section coincides with the TNB basis* $\mathcal{B}_f = (\mathbf{a}_t(s), \mathbf{a}_n(s), \mathbf{a}_b)$. Consequently, from equation [3.8], the unique non-zero component of the initial curvature vector is $\bar{\kappa} := \bar{\kappa}_3 = k$, so that $\bar{\mathbf{k}} = \bar{\kappa}\bar{\mathbf{a}}_3$.

In-plane and out-of-plane behavior

The equations of motion governing the planar beam can be derived from the general model in 3D by letting $\bar{\kappa}_1 = \bar{\kappa}_2 = 0$. It can be checked from these equations that, if the beam is loaded by external (or inertia) forces contained in the plane π, i.e. if $p_3 = c_1 = c_2 = 0$, then the model in 3D admits a *planar solution* $u_3 = \theta_1 = \theta_2 = 0$; accordingly, the beam assumes a current configuration still contained in π. If, in contrast, the loads act orthogonally to π, i.e. if $p_1 = p_2 = c_3 = 0$, then the solution is *complete*, in the sense that *all* the displacement and rotation components are different from zero. Therefore, out-of-plane forces call for in-plane displacements, so that the general model must be used to evaluate the response. In this section, therefore, we will formulate the planar model only, able to describe the uncoupled in-plane behavior.

3.3.1 Kinematics

Displacement, rotation and strains

The displacement is:

$$\mathbf{u} := u_1(s,t)\bar{\mathbf{a}}_1(s) + u_2(s,t)\bar{\mathbf{a}}_2(s) \qquad [3.68]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and the rotation is $\mathbf{R}(s,t)$, whose scalar representation in $\bar{\mathcal{B}}$ becomes:

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [3.69]$$

where $\theta := \theta_3$; therefore, any geometric transformations depend on three scalar fields, $u_1(s,t)$, $u_2(s,t)$, $\theta(s,t)$.

The geometric boundary conditions are of the type:

$$u_{1H} = \breve{u}_{1H}, \qquad u_{2H} = \breve{u}_{2H}, \qquad \theta_H = \breve{\theta}_H, \qquad H = A, B \qquad [3.70]$$

The strain vector [3.20] and the change of curvature vector [3.28] reduce to:

$$\begin{aligned} \mathbf{e} &= \varepsilon\bar{\mathbf{a}}_1 + \gamma\bar{\mathbf{a}}_2 \\ \boldsymbol{\chi} &= \chi\bar{\mathbf{a}}_3 \end{aligned} \qquad [3.71]$$

where $\gamma := \gamma_2$, $\chi := \chi_3$, and where the components follow from equations [3.32] and [3.35][8]:

$$\begin{aligned} \varepsilon &= -1 + (1 + u_1' - \bar{\kappa}u_2)\cos\theta + (u_2' + \bar{\kappa}u_1)\sin\theta \\ \gamma &= -(1 + u_1' - \bar{\kappa}u_2)\sin\theta + (u_2' + \bar{\kappa}u_1)\cos\theta \\ \chi &= \theta' \end{aligned} \qquad [3.72]$$

Velocity and spin

The velocity vector is obtained by time-differentiation of equation [3.68]; the spin vector follows from the specialization of either equation [3.39]:

$$\begin{aligned} \mathbf{v} &= \dot{u}_1(s,t)\bar{\mathbf{a}}_1 + \dot{u}_2(s,t)\bar{\mathbf{a}}_2 \\ \boldsymbol{\omega} &= \omega_3\mathbf{a}_3 = \dot{\theta}\mathbf{a}_3 \end{aligned} \qquad [3.73]$$

8. Alternatively, by direct calculations, we have $\bar{\mathbf{a}}_1 + \bar{\mathbf{e}} = \mathbf{R}^T(\bar{\mathbf{a}}_1 + \mathbf{u}')$ and, moreover, $\mathbf{u}' = u_1'\bar{\mathbf{a}}_1 + u_2'\bar{\mathbf{a}}_2 + \bar{\mathbf{k}} \times \mathbf{u}$; therefore:

$$\begin{pmatrix} 1+\varepsilon \\ \gamma \\ 0 \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 + u_1' - \bar{\kappa}u_2 \\ u_2' + \bar{\kappa}u_1 \\ 0 \end{pmatrix}$$

Concerning the curvature, since $\mathcal{S}$ and $\hat{\mathcal{S}}$ are planar curves, then $\bar{\mathbf{k}} = [\arccos(\bar{\mathbf{a}}_1 \cdot \mathbf{i}_1)]'\,\bar{\mathbf{a}}_3$ and $\mathbf{k} = \mathbf{k}_r \equiv \mathbf{k}_c = [\arccos(\mathbf{a}_1 \cdot \mathbf{i}_1)]'\,\bar{\mathbf{a}}_3$; hence $\boldsymbol{\chi} = \mathbf{k} - \bar{\mathbf{k}} = \theta'\bar{\mathbf{a}}_3$ with $\theta := \arccos(\mathbf{a}_1 \cdot \bar{\mathbf{a}}_1)$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Strain rates

The *strain-rates* $\dot{\mathbf{e}} = \dot{\varepsilon}\bar{\mathbf{a}}_1 + \dot{\gamma}_2\bar{\mathbf{a}}_2$ and $\dot{\boldsymbol{\chi}} = \dot{\chi}\bar{\mathbf{a}}_3$ admit the following components:

$$\begin{aligned}
\dot{\varepsilon} &= (\dot{u}_1' - \bar{\kappa}\dot{u}_2)\cos\theta + (\dot{u}_2' + \bar{\kappa}\dot{u}_1)\sin\theta + \gamma\dot{\theta} \\
\dot{\gamma} &= -(\dot{u}_1' - \bar{\kappa}\dot{u}_2)\sin\theta + (\dot{u}_2' + \bar{\kappa}\dot{u}_1)\cos\theta - (1+\varepsilon)\dot{\theta} \\
\dot{\chi} &= \dot{\theta}'
\end{aligned} \qquad [3.74]$$

where equation [3.72][9].

3.3.2 Dynamics

Let the beam be loaded by distributed in-plane forces $\mathbf{p} = \bar{p}_1\bar{\mathbf{a}}_1 + \bar{p}_2\bar{\mathbf{a}}_2$ and in-plane couples $\mathbf{c} = \bar{c}\bar{\mathbf{a}}_3$; moreover, let $\mathbf{P}_H = \bar{P}_{1H}\bar{\mathbf{a}}_1 + \bar{P}_{2H}\bar{\mathbf{a}}_2$ and $\mathbf{C}_H = \bar{C}_H\bar{\mathbf{a}}_3$ $(H = A, B)$, be forces and couples applied at the boundaries.

Balance equations

The balance equations, in vector form, still read as in equation [2.107]:

$$\begin{aligned}
&\mathbf{t}' + \mathbf{p} = m\ddot{\mathbf{u}} \\
&\mathbf{m}' + \mathbf{x}' \times \mathbf{t} + \mathbf{c} = \mathbf{I}_G\dot{\boldsymbol{\omega}}
\end{aligned} \qquad [3.75]$$

and boundary conditions as in equation [3.45]:

$$\begin{aligned}
&(\mathbf{P}_H \pm \mathbf{t}_H)\cdot\mathbf{v}_H = 0 \\
&(\mathbf{C}_H \pm \mathbf{m}_H)\cdot\boldsymbol{\omega}_H = 0, \qquad H = A, B
\end{aligned} \qquad [3.76]$$

9. In matrix form, it is $\dot{\boldsymbol{\varepsilon}} = \boldsymbol{D}(\boldsymbol{w}, \boldsymbol{w}')\,\dot{\boldsymbol{w}}$, where $\boldsymbol{\varepsilon} := (\varepsilon, \gamma, \chi)^T$, $\boldsymbol{w} := (u_1, u_2, \theta)^T$ and:

$$\boldsymbol{D}(\boldsymbol{w}, \boldsymbol{w}') := \begin{pmatrix} (\cos\theta)\,\partial_s + \bar{\kappa}\sin\theta & (\sin\theta)\,\partial_s - \bar{\kappa}\cos\theta & \gamma \\ -(\sin\theta)\,\partial_s + \bar{\kappa}\cos\theta & (\cos\theta)\,\partial_s + \bar{\kappa}\sin\theta & -(1+\varepsilon) \\ 0 & 0 & \partial_s \end{pmatrix}$$

is the *kinematic operator* of the curved planar beam. If it is evaluated at the reference configuration, it becomes the *infinitesimal kinematic operator* of the linear theory:

$$\boldsymbol{D}_0 := \boldsymbol{D}(\mathbf{0}, \mathbf{0}) = \begin{pmatrix} \partial_s & -\bar{\kappa} & 0 \\ \bar{\kappa} & \partial_s & -1 \\ 0 & 0 & \partial_s \end{pmatrix}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Now, however, the force-stress $\mathbf{t}$ and the couple-stress $\mathbf{m}$, are contained in π, i.e.:

$$\begin{aligned}\mathbf{t} &= N\mathbf{a}_1 + T\mathbf{a}_2 \\ \mathbf{m} &= M\mathbf{a}_3\end{aligned} \qquad [3.77]$$

where N is the axial force, $T := T_2$ is the shear force and $M := M_3$ is the bending moment; moreover, since $\dot{\boldsymbol{\omega}}$ is orthogonal to π, the only significant part of $\mathbf{I}_G$ is the mass moment of inertia with respect to $\mathbf{a}_3$, $I := I_3$.

By projecting the field equations onto the reference basis, we have[10, 11]:

$$\begin{aligned}&[N' - (\bar{\kappa} + \chi) T] \cos\theta - [T' + (\bar{\kappa} + \chi) N] \sin\theta + \bar{p}_1 - m\ddot{u}_1 = 0 \\ &[N' - (\bar{\kappa} + \chi) T] \sin\theta + [T' + (\bar{\kappa} + \chi) N] \cos\theta + \bar{p}_2 - m\ddot{u}_2 = 0 \\ &M' + (1 + \varepsilon) T - \gamma N + \bar{c} - I\ddot{\theta} = 0\end{aligned} \qquad [3.78]$$

By projecting, at the free ends, the boundary conditions, it follows:

$$\begin{aligned}&\mp (N\cos\theta - T\sin\theta)_H = \bar{P}_{1H} \\ &\mp (N\sin\theta + T\cos\theta)_H = \bar{P}_{2H} \\ &\mp M_H = \bar{C}_H, \qquad H = A, B\end{aligned} \qquad [3.79]$$

---

10. Indeed, by remembering the Poisson formula, and accounting for $\mathbf{k} = (\bar{\kappa} + \chi)\,\mathbf{a}_3$, we have:

$$\begin{aligned}\mathbf{t}' &= N'\mathbf{a}_1 + T'\mathbf{a}_2 + \mathbf{k} \times \mathbf{t} = \left[N' - (\bar{\kappa} + \chi) T\right] \mathbf{a}_1 + \left[T' + (\bar{\kappa} + \chi) N\right] \mathbf{a}_2 \\ \mathbf{m}' &= M'\mathbf{a}_3 + \mathbf{k} \times \mathbf{m} = M'\mathbf{a}_3\end{aligned}$$

Moreover, since, $\mathbf{x}' = \mathbf{R}\,(\bar{\mathbf{a}}_1 + \mathbf{e})$, then:

$$\mathbf{x}' \times \mathbf{t} = [(1 + \varepsilon)\,\mathbf{a}_1 + \gamma\mathbf{a}_2] \times (N\mathbf{a}_1 + T\mathbf{a}_2) = [(1 + \varepsilon)\,T - \gamma N]\,\mathbf{a}_3$$

Finally, $\mathbf{a}_i = \mathbf{R}\bar{\mathbf{a}}_i$ must be used.

11. In matrix form, $\boldsymbol{D}^\star(\boldsymbol{w}, \boldsymbol{w}')\,\boldsymbol{\sigma} = \boldsymbol{p}$, where $\boldsymbol{\sigma} := (N, T, M)^T$, $\boldsymbol{p} := (\bar{p}_1, \bar{p}_2, \bar{c})^T$ and:

$$\boldsymbol{D}^\star\left(\boldsymbol{w}, \boldsymbol{w}'\right) := \begin{pmatrix} (-\cos\theta)\,\partial_s + (\bar{\kappa} + \chi)\sin\theta & [(\sin\theta)\,\partial_s + (\bar{\kappa} + \chi)\cos\theta] & 0 \\ (-\sin\theta)\,\partial_s - (\bar{\kappa} + \chi)\cos\theta & (-\cos\theta)\,\partial_s + (\bar{\kappa} + \chi)\sin\theta & 0 \\ \gamma & -(1 + \varepsilon) & -\partial_s \end{pmatrix}$$

is the *equilibrium operator* of the curved planar beam; it is the adjoint of the kinematic operator $\boldsymbol{D}(\boldsymbol{w}, \boldsymbol{w}')$. When, in the framework of the linear theory, it is evaluated at the reference configuration, it reduces to adjoint of $\boldsymbol{D}_0$, namely:

$$\boldsymbol{D}_0^\star := \boldsymbol{D}^\star(\mathbf{0}, \mathbf{0}) = \begin{pmatrix} -\partial_s & \bar{\kappa} & 0 \\ -\bar{\kappa} & -\partial_s & 0 \\ 0 & -1 & -\partial_s \end{pmatrix}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

3.3.3 The Virtual Power Principle

The balance equations can, of course, be derived by specializing the VPP for the planar case and restarting the procedure. We will show this approach as an exercise.

The Virtual Power equation, $\mathcal{P}_{ext} = \mathcal{P}_{int}$, when vectors in equation [3.41] are expressed in the reference basis, specializes as follows:

$$\int_{\mathcal{S}} (\bar{p}_1 v_1 + \bar{p}_2 v_2 + \bar{c}\omega)\, ds + \sum_{H=A}^{B} \left(\bar{P}_{1H} v_{1H} + \bar{P}_{2H} v_{2H} + \bar{C}_H \omega_H\right)$$
$$= \int_{\mathcal{S}} (N\dot{\varepsilon} + T\dot{\gamma} + M\dot{\chi})\, ds, \qquad \forall \left(\dot{u}_1, \dot{u}_2, \dot{\theta}\right) \tag{3.80}$$

Accordingly, if the Strain-rate-velocity relationships [3.74] are introduced in this equation, and $v_1 = \dot{u}_1$, $v_2 = \dot{u}_2$, $\omega = \dot{\theta}$ are used, the principle becomes:

$$\int_{\mathcal{S}} \left(\bar{p}_1 \dot{u}_1 + \bar{p}_2 \dot{u}_2 + \bar{c}\dot{\theta}\right) ds + \sum_{H=A}^{B} \left(\bar{P}_{1H} \dot{u}_{1H} + \bar{P} \dot{u}_{2H} + \bar{C}_H \dot{\theta}_H\right)$$
$$= \int_{\mathcal{S}} \left\{ N \left[\left(\dot{u}_1' - \bar{\kappa}\dot{u}_2\right) \cos\theta + \left(\dot{u}_2' + \bar{\kappa}\dot{u}_1\right) \sin\theta + \gamma\dot{\theta} \right] \right. \tag{3.81}$$
$$\left. + T \left[- \left(\dot{u}_1' - \bar{\kappa}\dot{u}_2\right) \sin\theta + \left(\dot{u}_2' + \bar{\kappa}\dot{u}_1\right) \cos\theta - (1+\varepsilon)\dot{\theta} \right] + M\dot{\theta}' \right\} ds$$

or, after an integration by parts:

$$\int_{\mathcal{S}} \left\{ \left[(N\cos\theta - T\sin\theta)' - \bar{\kappa}N\sin\theta - \bar{\kappa}T\cos\theta + \bar{p}_1 \right] \dot{u}_1 \right.$$
$$+ \left[- (N\sin\theta + T\cos\theta)' + \bar{\kappa}N\cos\theta - \bar{\kappa}T\sin\theta + \bar{p}_2 \right] \dot{u}_2$$
$$\left. + \left[M' - N\gamma + T(1+\varepsilon) + \bar{c} \right] \dot{\theta} \right\} ds + \{[- (N\cos\theta - T\sin\theta) \tag{3.82}$$
$$+ \bar{P}_1] \, \dot{u}_1\}_A^B + \{[- (N\sin\theta + T\cos\theta) + \bar{P}_2] \, \dot{u}_2\}_A^B + \left\{ -[M + \bar{C}]\dot{\theta} \right\}_A^B = 0$$

Finally, since $(\sin\theta)' = \chi\cos\theta$, $(\cos\theta)' = -\chi\sin\theta$, equations [3.78] are reobtained, together with equation [3.79] on the free boundary.

3.3.4 Constitutive law

The uncoupled linear elastic law becomes:

$$N = \mathring{N} + EA\varepsilon$$
$$T = \mathring{T} + GA_s\gamma \tag{3.83}$$
$$M = \mathring{M} + EJ\chi$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where $GA_s := GA_2$ is the shear stiffness, $EJ := EJ_3$ is the bending stiffness and overmarked quantities are prestresses.

3.3.5 Fundamental Problem

The equations governing the motion are obtained by combining the strain–displacement relationships [3.72], the balance equations [3.78] and the constitutive law [3.83], together with boundary conditions [3.70] and [3.79].

The exact equations for unprestressed beams

By ignoring the prestress, the following exact equations of motion are derived:

$$\begin{aligned}
&[\{EA[(1+u_1'-\bar{\kappa}u_2)\cos\theta+(u_2'+\bar{\kappa}u_1)\sin\theta-1]\}' \\
&\quad-(\bar{\kappa}+\chi)\,GA_s[(u_2'+\bar{\kappa}u_1)\cos\theta-(1+u_1'-\bar{\kappa}u_2)\sin\theta]]\cos\theta \\
&\quad-[\{GA_s[(u_2'+\bar{\kappa}u_1)\cos\theta-(1+u_1'-\bar{\kappa}u_2)\sin\theta]\}' \\
&\quad+(\bar{\kappa}+\chi)\,EA[(1+u_1'-\bar{\kappa}u_2)\cos\theta+(u_2'+\bar{\kappa}u_1)\sin\theta-1]]\sin\theta \\
&\quad+\bar{p}_1-m\ddot{u}_1=0 \\
&[\{EA[(1+u_1'-\bar{\kappa}u_2)\cos\theta+(u_2'+\bar{\kappa}u_1)\sin\theta-1]\}' \\
&\quad-(\bar{\kappa}+\chi)\,GA_s[(u_2'+\bar{\kappa}u_1)\cos\theta-(1+u_1'-\bar{\kappa}u_2)\sin\theta]]\sin\theta \\
&\quad+[\{GA_s[(u_2'+\bar{\kappa}u_1)\cos\theta-(1+u_1'-\bar{\kappa}u_2)\sin\theta]\}' \\
&\quad+(\bar{\kappa}+\chi)\,EA[(1+u_1'-\bar{\kappa}u_2)\cos\theta+(u_2'+\bar{\kappa}u_1)\sin\theta-1]]\cos\theta \\
&\quad+\bar{p}_2-m\ddot{u}_2=0 \\
&\{EJ\theta'\}'+(GA_s-EA)[(1+u_1'-\bar{\kappa}u_2)\cos\theta+(u_2' \\
&\quad+\bar{\kappa}u_1)\sin\theta][(u_2'+\bar{\kappa}u_1)\cos\theta-(1+u_1'-\bar{\kappa}u_2)\sin\theta] \\
&\quad-GA_s[(u_2'+\bar{\kappa}u_1)\cos\theta-(1+u_1'-\bar{\kappa}u_2)\sin\theta]+\bar{c}-I\ddot{\theta}=0
\end{aligned} \qquad [3.84]$$

with the mechanical boundary conditions:

$$\begin{aligned}
&\mp\{EA[(1+u_1'-\bar{\kappa}u_2)\cos\theta+(u_2'+\bar{\kappa}u_1)\sin\theta-1]\cos\theta \\
&\qquad-GA_s[(u_2'+\bar{\kappa}u_1)\cos\theta-(1+u_1'-\bar{\kappa}u_2)\sin\theta]\sin\theta\}_H=\bar{P}_{1H} \\
&\mp\{EA[(1+u_1'-\bar{\kappa}u_2)\cos\theta+(u_2'+\bar{\kappa}u_1)\sin\theta-1]\sin\theta \\
&\qquad+GA_s[(u_2'+\bar{\kappa}u_1)\cos\theta-(1+u_1'-\bar{\kappa}u_2)\sin\theta]\cos\theta\}_H=\bar{P}_{2H} \\
&\mp\{EJ\theta'\}_H=\bar{C}_H, \qquad\qquad H=A,B
\end{aligned} \qquad [3.85]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and the geometric boundary conditions:

$$u_{1H} = \breve{u}_{1H}, \qquad u_{2H} = \breve{u}_{2H}, \qquad \theta_H = \breve{\theta}_H, \qquad H = A, B \tag{3.86}$$

Linearized theory for prestressed beams

Equations of motions for prestressed beams are obtained here in the framework of the linearized theory.

The strain–displacement relationships [3.72] become:

$$\begin{aligned} \varepsilon &= u_1' - \bar{\kappa} u_2 \\ \gamma &= u_2' + \bar{\kappa} u_1 - \theta \\ \kappa &= \theta' \end{aligned} \tag{3.87}$$

The incremental balance equations are obtained from equation [3.78], once all the quantities are split into a pre-existing (large) part and a (small) incremental part, namely $\bar{p}_i = \mathring{p}_i + \tilde{p}_i$, $\bar{c} = \mathring{c} + \tilde{c}$ for forces and $N = \mathring{N} + \tilde{N}$, $T = \mathring{T} + \tilde{T}$, $M = \mathring{M} + \tilde{M}$ for stresses. By retaining first-order terms only and accounting for the pre-existing equilibrium, the balance equations transform into:

$$\begin{aligned} &\tilde{N}' - \bar{\kappa}\tilde{T} - (\theta\mathring{T})' - \bar{\kappa}\theta\mathring{N} + \tilde{p}_1 = m\ddot{u}_1 \\ &\tilde{T}' + \bar{\kappa}\tilde{N} + (\theta\mathring{N})' - \bar{\kappa}\theta\mathring{T} + \tilde{p}_2 = m\ddot{u}_2 \\ &M' + \tilde{T} + (u_1' - \bar{\kappa}u_2)\mathring{T} - (u_2' + \bar{\kappa}u_1 - \theta)\mathring{N} + \tilde{p} = I\ddot{\theta} \end{aligned} \tag{3.88}$$

By the same procedure, the boundary conditions follow from equation [3.79]:

$$\begin{aligned} &\tilde{N}' - \bar{\kappa}\tilde{T} - (\theta\mathring{T})' - \bar{\kappa}\theta\mathring{N} + \tilde{p}_1 = m\ddot{u}_1 \\ &\tilde{T}' + \bar{\kappa}\tilde{N} + (\theta\mathring{N})' - \bar{\kappa}\theta\mathring{T} + \tilde{p}_2 = m\ddot{u}_2 \\ &M' + \tilde{T} + (u_1' - \bar{\kappa}u_2)\mathring{T} - (u_2' + \bar{\kappa}u_1 - \theta)\mathring{N} + \tilde{p} = I\ddot{\theta} \end{aligned} \tag{3.89}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By using the elastic law [3.83], the equations of motion are finally derived[12]:

$$
\begin{aligned}
&[EA(u_1' - \bar{\kappa} u_2)]' - \bar{\kappa} GA_s(u_2' + \bar{\kappa} u_1 - \theta) - (\theta \mathring{T})' - \bar{\kappa}\theta \mathring{N} + \tilde{p}_1 - m\ddot{u}_1 = 0 \\
&[GA_s(u_2' + \bar{\kappa} u_1 - \theta)]' + \bar{\kappa} EA(u_1' - \bar{\kappa} u_2) + (\theta \mathring{N})' - \bar{\kappa}\theta \mathring{T} + \tilde{p}_2 - m\ddot{u}_2 = 0 \\
&(EJ\theta')' + GA_s(u_2' + \bar{\kappa} u_1 - \theta) + (u_1' - \bar{\kappa} u_2)\mathring{T} - (u_2' + \bar{\kappa} u_1 - \theta)\mathring{N} + \tilde{c} - I\ddot{\theta} = 0
\end{aligned}
\qquad [3.90]
$$

Similarly, the mechanical boundary conditions are:

$$
\begin{aligned}
&\mp [EA(u_1' - \bar{\kappa} u_2) - \theta \mathring{T}]_H = \tilde{P}_{1H} \\
&\mp [GA_s(u_2' + \bar{\kappa} u_1 - \theta) + \theta \mathring{N}]_H = \tilde{P}_{2H} \\
&\mp [EJ\theta']_H = \tilde{C}_H
\end{aligned}
\qquad [3.91]
$$

If the prestress is ignored (or it is absent), the model reduces to the linear Timoshenko beam.

3.4 Summary

Here, we summarize the main results of this chapter. We formulated a model of curved beam immersed in a 3D-space, made up of a flexible centerline and rigid cross-sections, modeled as a polar, one-dimensional continuum.

12. They are of the type $\boldsymbol{Lu} + \boldsymbol{Gu} = \tilde{\boldsymbol{p}}$ in the field, and $\boldsymbol{\mathcal{L}}_H \boldsymbol{u} + \boldsymbol{\mathcal{G}}_H \boldsymbol{u} = \tilde{\boldsymbol{P}}$ on the boundary (equation [1.72]), where:

$$
\boldsymbol{L} := \begin{bmatrix} -EA\partial_s^2 + \bar{\kappa}^2 GA_s & \bar{\kappa}' EA + \bar{\kappa} EA\partial_s + \bar{\kappa} GA_s\partial_s & \bar{\kappa} GA_s \\ -\bar{\kappa}' GA_s - \bar{\kappa} GA_s\partial_s - \bar{\kappa} EA\partial_s & -GA_s\partial_s^2 + \bar{\kappa}^2 EA & GA_s\partial_s \\ -\bar{\kappa} GA_s & -GA_s\partial_s & GA_s - EJ\partial_s^2 \end{bmatrix}
$$

$$
\boldsymbol{G} := \begin{bmatrix} 0 & 0 & \mathring{T}' + \mathring{T}\partial_s + \bar{\kappa}\mathring{N} \\ 0 & 0 & -\mathring{N}' - \mathring{N}\partial_s + \bar{\kappa}\mathring{T} \\ -\mathring{T}\partial_s + \bar{\kappa}\mathring{N} & \mathring{N}\partial_s + \bar{\kappa}\mathring{T} & -\mathring{N} \end{bmatrix}
$$

are elastic and geometric stiffness operators, and:

$$
\boldsymbol{\mathcal{L}}_H := \begin{bmatrix} EA\partial_s & -\bar{\kappa} EA & 0 \\ \bar{\kappa} GA_s & GA_s\partial_s & -GA_s \\ 0 & 0 & EJ\partial_s \end{bmatrix}_H , \quad \boldsymbol{\mathcal{G}}_H := \begin{bmatrix} 0 & 0 & -\mathring{T} \\ 0 & 0 & \mathring{N} \\ 0 & 0 & 0 \end{bmatrix}_H
$$

are their counterpart on the free boundary. They hold in case of uniform initial curvature $\bar{\kappa}$, uniform elastic coefficients EA, GA_s and uniform prestresses $\mathring{N}$, $\mathring{T}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

We first described the reference configuration. We introduced the parametric equations of the centerline and the *deviation angle*, which measures the amplitude of the rotation about the tangent to the beam axis, which leads the TNB (Frenet) triad, intrinsic to the centerline, to match the principal inertia triad, intrinsic to the cross-section. The deviation angle only alters the torsion of the curve and changes the basis on which the principal curvature vector is projected.

In analyzing kinematics, we mainly stressed the differences existing within the straight beam model. Indeed, current and reference curvatures measure the current state of the beam, that, however, is affected by the initial curvature existing in the reference configuration. Strain is then introduced as a *change of curvature*, which is conveniently defined as the difference between present and initial curvatures, both related to the reference configuration. To get scalar components, however, we had to take into account that the reference basis is not fixed, but depends on the abscissa; therefore, the space-derivative of the rotation tensor is *not* the derivative of the components. A formula was derived to perform calculations, simply by expressing the tensor in a basis of tensors, and then performing differentiations of products. About kinetic quantities, which involve time-derivatives, we noted that, being the initial curvature independent of time, this is ineffective, so that kinetic magnitudes assume the same expressions they have for the straight beam.

Dynamics is ruled by the VPP (or momentum principles), which, in vector form, has the same expression holding for the straight beam. However, when the balance equations are projected onto the curved reference basis, the scalar equations contain the initial curvature components.

Constitutive equations for linear hyperelastic materials were recalled, identical to those for the straight beam, except for the appearance of the change of curvature instead of the present curvature.

The Fundamental Problem for the curved beam is governed by 18 equations in 18 unknowns. For the unprestressed case, we derived exact equations of motion for the unknown displacements. Linearized equations, governing the motion around a prestressed configuration were also obtained.

Finally, the special case of *planar beam* was analyzed. Here, the TNB and inertia bases coincide and the initial curvature is described by a scalar curvature only. An explicit form for the exact motion of the equations was given in absence of prestress, and linearized equations for prestress.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Chapter 4

Internally Constrained Beams

In this chapter, we analyze several models of internally constrained beam, aimed (a) at illustrating the procedures discussed for the metamodel, and (b) at introducing models useful for engineering applications. First, we perform an order of magnitude analysis, based on an energy criterion, to predict, on the grounds of the geometric characteristics of the beam, when an internally constrained model can be adopted. Then, by following the methods illustrated in Chapter 1 (namely the mixed, the displacement and the hybrid formulations), we derive models for the following straight beams: (a) unshearable (Euler–Bernoulli), (b) inextensible and unshearable (Euler's elastica), (c) untwistable, inextensible and unshearable (compact-section or boxed Euler's elastica), (d) foil-beam (one-plane-inflexible Euler's elastica), (e) shear–shear beam (homogenized tower-building). Finally, we develop a model for a (f) curved planar beam, inextensible and unshearable.

4.1 Stiff beams and internal constraints

In Chapters 2 and 3 we formulated models of straight and curved beams, directly derived as one-dimensional polar continua. Since we did *not* introduce any restrictions to kinematics, these models should be considered as *internally unconstrained one-dimensional models*[1]. On the other hand, in section 1.3, we introduced the concept of an *internally constrained metamodel*, as an ideal beam in which one or more (so-called

1. Of course, if we had derived the models from a 3D-continuum, by enforcing rigidity of the sections, we should have referred to them as "internally constrained three-dimensional models".

Mathematical Models of Beams and Cables Angelo Luongo and Daniele Zulli
© 2013 ISTE Ltd. Published 2013 by ISTE Ltd.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

constrained) strains identically vanish in the domain. Since internal constraints reduce the number of the unknowns of the Fundamental Problem, we are now interested in formulating specific models of constrained beams.

Order of magnitude analysis

The mechanical experience tells us that a beam can more easily be bent, rather than extended; similarly, it can more easily be twisted if its cross-section is thin and open, rather than closed or compact. Such qualitative considerations suggest to us to neglect some of the strains and formulate (partially) rigid models. However, in order to derive a reliable internally constrained model, we need a criterion to evaluate *a priori* if a strain is or is not negligible with respect to the others. The best way to proceed is not to directly compare strains among them, but rather to compare their contribution to the elastic energy. Indeed, we could have small strains that, when multiplied by large elastic coefficients, could significantly participate in the response of the beam. In contrast, by referring to energy, we have a global measure accounting for strains and stiffness, thus being unaffected by this drawback. Of course, the simplest case of diagonal elastic law strongly simplifies the analysis, and therefore we are limited to this law.

With these ideas in mind, we rewrite the quadratic elastic potential (equation [2.163], for the unprestressed beam, and express it in terms of stresses, instead of strains (also known as *complementary* elastic energy):

$$\begin{aligned}\phi &= \frac{1}{2}\left(EA\varepsilon^2 + GA_2\gamma_2^2 + GA_3\gamma_3^2 + GJ_1\kappa_1^2 + EJ_2\kappa_2^2 + EJ_3\kappa_3^2\right)\\ &= \underbrace{\frac{1}{2}\frac{N^2}{EA}}_{=:\phi_e} + \underbrace{\frac{1}{2}\left(\frac{T_2^2}{GA_2} + \frac{T_3^2}{GA_3}\right)}_{=:\phi_s} + \underbrace{\frac{1}{2}\frac{M_1^2}{GJ_1}}_{=:\phi_t} + \underbrace{\frac{1}{2}\left(\frac{M_2^2}{EJ_2} + \frac{M_3^2}{EJ_3}\right)}_{=:\phi_f}\end{aligned} \qquad [4.1]$$

The four addenda constitute the *extensional*, *shear*, *torsional* and *flexural* contributions to the elastic potential, respectively. Our goal is to perform an order of magnitude analysis of the different contributions in quite common situations.

First, we observe that when forces $\mathrm{O}(P)$ are applied at one end of a beam, then $N = \mathrm{O}(P)$, $T_j = \mathrm{O}(P)$, $M_1 = \mathrm{O}(Pr)$, $M_i = \mathrm{O}(Pl)$ with $i = 2, 3$, where r is a characteristic linear dimension of the cross-section and l is the length of the beam[2]. From de Saint-Venant theory and geometric properties of planar figures, we know that: $A_i = \varsigma_i A$, where ς_i is the *shear factor*, usually $\mathrm{O}(1)$; if the cross-section is compact, or boxed, $J_1 = \mathrm{O}(J_i)$ and moreover, $J_i = \mathrm{O}(r^2 A)$, since the radii of inertia

2. Here, and in the following, O is the Landau symbol, to be read as “Capital o of”, meaning “of the same order of”.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

are of the order of the characteristic dimension of the cross-section. However, if the beam is thin-walled, and its section is *open*, then $J_1 = O(rb^3)$, $J_i = O(rb^3)$, with $b \ll r$ being a characteristic thickness. A special case is represented by foil-beams, for which, when $\mathbf{a}_3$ is the "strong" axis, $J_1 = O(rb^3)$, $J_2 = O(rb^3)$, $J_3 = O(r^3b)$. In all cases, $G = O(E)$.

By comparing shear and flexural energies, we have $\phi_s/\phi_f = O\left(r^2/\varsigma_i l^2\right)$. Since, typically, the *slenderness ratio* l/r ranges between 20 and 100, the shear contribution is usually negligible. It can be of little importance only in short beams (in which geometrical nonlinearities have small effects) or for very shear-deformable beams (i.e. when $\varsigma_i \ll 1$). This case occurs, for example, in beam-like structures, such as Vierendel or shear-type frames, when we want to homogenize them as beams. In limit cases, the shear contribution can dominate the flexural contribution; as a result, the beam is called a *shear-beam*. Except for these particular cases, the beams are usually assumed *unshearable*. It should be observed that consistently with the no-shear assumption, when we tackle dynamic problems, *also the flexural rotatory inertia should be neglected*, unless large non-structural masses are solid with the beam[3].

By comparing extensional and flexural energies, we have $\phi_e/\phi_f = O\left(r^2/l^2\right) \ll 1$. Since the extensional contribution to the energy is negligible, a model of *inextensible beam* can be adopted. However, care must be taken in neglecting extension, since such kind of model is geometrically incompatible with external constraints that prevent *both* longitudinal displacements at the ends, u_{1A}, u_{1B}. Therefore, the dynamics of planar beams with longitudinally movable or immovable ends are very different.

If we compare torsional and flexural contributions for compact or boxed beams, we find $\phi_t/\phi_f = O\left(r^2/l^2\right) \ll 1$. Such beams, therefore, are almost *untwistable*, and a simplified no-twist model can be adopted. However, this is a consequence of having considered external forces acting with small eccentricity with respect to the beam axis, as usually happens. If, in contrast, torques $C_1 = O(Pl)$ are applied to the beam, twist cannot be neglected. If we refer, instead, to open thin-walled beams, then $\phi_t/\phi_f = O\left(r^4/b^2l^2\right)$, which is usually of order 1, if $b/r = O(r/l)$. Therefore, such beams are twistable.

The special case of the foil-beams is now addressed. Because of the large ratio between the two inertia, the ratio between the two contributions to the flexural energy is $\phi_{f_3}/\phi_{f_2} = O\left(b^2/r^2\right) \ll 1$. Therefore, an approximated model in which the

3. As a matter of fact, the non-dimensional ratio between inertia couples and forces is $c_3^{in}/lp_2^{in} = O\left(\left(I_3\ddot{\theta}_3\right)/\left(lm\ddot{u}_2\right)\right)$. By assuming harmonic motion and $\theta_3 = O(u_2/l)$, consistent with no-shear hypothesis (see equation [4.24]), the previous ratio is $O\left(r^2/l^2\right)$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

bending around the “strong” axis is ignored can be adopted. In contrast, since $\phi_t/\phi_{f_2} = \mathrm{O}(r/l)$, i.e. the ratio is less than 1 but not so small, twist cannot be neglected.

4.2 The general approach

To formulate an internally constrained model, we will closely follow the methods discussed in sections 1.3.1 and 1.3.2, by referring to the beam metamodel. First, we have to realize that we can no longer use the vector approach, since strains, and therefore displacements, are generally *only partially* constrained. For example, if the beam is unshearable, only two components of the strain vector $\mathbf{e}$ are zero (namely γ_2, γ_3), not $\mathbf{e}$ itself, since $\varepsilon \neq 0$. Therefore, using the scalar form is mandatory. However, in order to avoid writing very long formulas, we will adopt a matrix notation, that, as we noted in section 2.2.5, naturally leads to Lagrangian balance equations.

Kinematics

Strains for the unconstrained model, in matrix form, are (equation [2.182]):

$$\begin{aligned} &\boldsymbol{e} = \boldsymbol{R}^T\left(\bar{\boldsymbol{a}}_1 + \boldsymbol{u}'\right) - \bar{\boldsymbol{a}}_1 \\ &\boldsymbol{k} = \bar{\boldsymbol{B}}_\omega \boldsymbol{\theta}' \end{aligned} \qquad [4.2]$$

where:

$$\boldsymbol{u} := \left(u_1, u_2, u_3\right)^T, \quad \boldsymbol{\theta} := \left(\theta_1, \theta_2, \theta_3\right)^T, \quad \boldsymbol{e} := \left(\varepsilon, \gamma_2, \gamma_3\right)^T, \quad \boldsymbol{k} := \left(\kappa_1, \kappa_2, \kappa_3\right)^T \qquad [4.3]$$

Moreover $\bar{\boldsymbol{a}}_1 := (1,0,0)^T$, $\boldsymbol{R}(\boldsymbol{\theta})$ is the rotation matrix defined by equation [2.6], and $\bar{\boldsymbol{B}}_\omega(\boldsymbol{\theta})$ is the spin-basis matrix given by equation [2.73].

Geometric boundary conditions are expressed by equation [2.19]:

$$\boldsymbol{u} = \breve{\boldsymbol{u}}, \qquad \boldsymbol{\theta} = \breve{\boldsymbol{\theta}} \qquad [4.4]$$

Strain-rates could be obtained by time-differentiation of the strains [4.2]; they, however, are more conveniently expressed in the form of equation [2.88]:

$$\begin{aligned} &\dot{\boldsymbol{e}} = \boldsymbol{R}^T \dot{\boldsymbol{u}}' + \boldsymbol{\Lambda} \boldsymbol{R}^T \bar{\boldsymbol{B}}_\omega \dot{\boldsymbol{\theta}} \\ &\dot{\boldsymbol{k}} = \boldsymbol{R}^T\left(\bar{\boldsymbol{B}}_\omega' \dot{\boldsymbol{\theta}} + \bar{\boldsymbol{B}}_\omega \dot{\boldsymbol{\theta}}'\right) \end{aligned} \qquad [4.5]$$

where the skew-symmetric matrix $\boldsymbol{\Lambda}$ is defined in equation [2.85].

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Constraints enforcing the vanishing of one or more of the strain components are considered. This entails that strain-rates and displacement-rates are restrained too.

Dynamics

The balance equations are provided by the Virtual Power Principle (VPP). In extended scalar form, it becomes:

$$\begin{aligned}&\int_{\mathcal{S}} \left(N\dot{\varepsilon} + T_2\dot{\gamma}_2 + T_3\dot{\gamma}_3 + M_1\dot{\kappa}_1 + M_2\dot{\kappa}_2 + M_3\dot{\kappa}_3\right) ds \\ &\quad = \int_{\mathcal{S}} \sum_{j=1}^{3} \left(\bar{p}_j v_j + \bar{c}_j \bar{\omega}_j\right) ds + \sum_{H=A}^{B} \sum_{j=1}^{3} \left[\bar{P}_j v_j + \bar{C}_j \bar{\omega}_j\right]_H\end{aligned} \quad [4.6]$$

or, in more compact form (see equation [2.143]):

$$\int_{\mathcal{S}} \left(\boldsymbol{t}^T \dot{\boldsymbol{e}} + \boldsymbol{m}^T \dot{\boldsymbol{k}}\right) ds = \int_{\mathcal{S}} \left(\bar{\boldsymbol{p}}^T \boldsymbol{v} + \bar{\boldsymbol{c}}^T \bar{\boldsymbol{\omega}}\right) ds + \sum_{H=A}^{B} \left[\bar{\boldsymbol{P}}^T \boldsymbol{v} + \bar{\boldsymbol{C}}^T \bar{\boldsymbol{\omega}}\right]_H \quad [4.7]$$

where:

$$\begin{aligned}&\bar{\boldsymbol{\omega}} := \left(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3\right)^T \\ &\boldsymbol{t} := \left(N, T_2, T_3\right)^T, \qquad \boldsymbol{m} := \left(M_1, M_2, M_3\right)^T \\ &\bar{\boldsymbol{p}} := \left(\bar{p}_1, \bar{p}_2, \bar{p}_3\right)^T, \qquad \bar{\boldsymbol{c}} := \left(\bar{c}_1, \bar{c}_2, \bar{c}_3\right)^T \\ &\bar{\boldsymbol{P}}_H := \left(\bar{P}_1, \bar{P}_2, \bar{P}_3\right)_H^T, \quad \bar{\boldsymbol{C}}_H := \left(\bar{C}_1, \bar{C}_2, \bar{C}_3\right)_H^T\end{aligned} \quad [4.8]$$

By relating the spin column matrix $\bar{\boldsymbol{\omega}}$ to the Tait-Bryan angles rate $\dot{\boldsymbol{\theta}}$ via $\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{B}}_\omega \dot{\boldsymbol{\theta}}$ (equation [2.71a]), and using $\boldsymbol{v} = \dot{\boldsymbol{u}}$, the VPP is finally written as:

$$\int_{\mathcal{S}} \left(\dot{\boldsymbol{e}}^T \boldsymbol{t} + \dot{\boldsymbol{k}}^T \boldsymbol{m}\right) ds = \int_{\mathcal{S}} \left(\dot{\boldsymbol{u}}^T \bar{\boldsymbol{p}} + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}}\right) ds + \sum_{H=A}^{B} \left[\dot{\boldsymbol{u}}^T \bar{\boldsymbol{P}} + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{C}}\right]_H \quad [4.9]$$

To obtain balance equations, we can proceed in three different ways according to the following:

1) *The mixed formulation* (see section 1.3.1). Here, the constraints are considered as auxiliary conditions restraining the field of the virtual motions in the VPP. Therefore, constraints are incorporated in the VPP via Lagrange multipliers. This approach does not alter the number of balance equations, which now contain active as well as reactive stresses.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2) *The displacement formulation* (see section 1.3.2). In this approach, a subset of displacements is expressed as *slave* of the remaining *master* displacements and eliminated by the VPP. In this way, a lower number of condensed balance equations are obtained, only involving active stresses.

3) *The hybrid formulation*. This is a combination of the two previous methods, in which a part of the constraints is accounted for by condensation of the slave displacements, and the remaining part via Lagrange multipliers. Therefore, the balance equations contain the active stresses and that part of the reactive stresses which spends (zero) power on the constrained strains appended to the integral principle.

In handling the virtual power equation, we will have to perform integration by parts. According to the mixed formulation, one integration is necessary to free the configuration variables $\boldsymbol{u}$, $\boldsymbol{\theta}$ by the space-derivatives. It is important to stress that this is exactly what we did in dealing with the unconstrained model. When, in contrast, the displacement (or hybrid) formulation is used, two integrations are necessary. As we mentioned in section 1.3.2, it is conceptually clearer to perform this operation in two steps: (a) we first substitute the strain rates, and, *by ignoring the constraints on the velocities*, we perform an integration by parts, aimed to free *all* (master and slave) velocities from space-derivatives; then, (b) we *substitute the velocity constraints* under the integral sign, and perform a new integration by parts to free the (remaining) *master* velocities from space derivatives.

The examples that follow clarify the procedure.

Constitutive law

We will consider a linear elastic beam, possibly prestressed, obeying, *when unconstrained*, the Hooke law [2.156]:

$$\begin{aligned} \boldsymbol{t} &= \mathring{\boldsymbol{t}} + \boldsymbol{E}_{ee}\boldsymbol{e} + \boldsymbol{E}_{ek}\boldsymbol{k} \\ \boldsymbol{m} &= \mathring{\boldsymbol{m}} + \boldsymbol{E}_{ke}\boldsymbol{k} + \boldsymbol{E}_{kk}\boldsymbol{k} \end{aligned} \qquad [4.10]$$

in which $\mathring{\boldsymbol{t}}$, $\mathring{\boldsymbol{m}}$ are prestresses and $\boldsymbol{E}_{\alpha\beta}$ are elastic submatrices. When constraints, are introduced, however, the stresses dual of the constrained strains becomes reactive, so that relevant constitutive law must be suppressed (see sections 1.3.1 and 1.5).

4.3 The unshearable straight beam in 3D

We develop a model for an unshearable straight beam, also called the *Euler–Bernoulli beam*[4]. We say that a straight beam is unshearable when the shear

4. Sometimes also called the *extensible* Euler's elastica.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

strains, γ_2, γ_3, identically vanish. By using equation [4.2], and accounting for $\varepsilon = \bar{\boldsymbol{a}}_1^T \boldsymbol{e}$ and $\gamma_2 = \bar{\boldsymbol{a}}_2^T \boldsymbol{e}$, $\gamma_3 = \bar{\boldsymbol{a}}_3^T \boldsymbol{e}$, the unconstrained (admissible) strains are as follows:

$$\begin{aligned} &\varepsilon = \bar{\boldsymbol{a}}_1^T \boldsymbol{R}^T \left(\bar{\boldsymbol{a}}_1 + \boldsymbol{u}'\right) - 1 \\ &\boldsymbol{k} = \bar{\boldsymbol{B}}_\omega \boldsymbol{\theta}' \end{aligned} \qquad [4.11]$$

and the constraints are:

$$\bar{\boldsymbol{a}}_2^T \boldsymbol{R}^T \left(\bar{\boldsymbol{a}}_1 + \boldsymbol{u}'\right) = 0, \qquad \bar{\boldsymbol{a}}_3^T \boldsymbol{R}^T \left(\bar{\boldsymbol{a}}_1 + \boldsymbol{u}'\right) = 0 \qquad [4.12]$$

We approach the problem either by the mixed or by the displacement formulations.

4.3.1 The mixed formulation

Since $\gamma_2 = 0$, $\gamma_3 = 0$ at any time, then $\dot{\gamma}_2 = 0$, $\dot{\gamma}_3 = 0$, so that the internal power $T_2\dot{\gamma}_2 + T_3\dot{\gamma}_3$, associated with the reactive shear forces, disappears from equation [4.6]. However, unshearability also constitutes a constraint for the velocity field, which must be accounted for when the principle is applied. The easiest way to accomplish the task is using Lagrangian multipliers. Accordingly, the VPP is written as a modification of equation [4.9]:

$$\begin{aligned} \int_{\mathcal{S}} \left(N\dot{\varepsilon} + \boldsymbol{m}^T \dot{\boldsymbol{k}}\right) ds &= \int_{\mathcal{S}} \left(\dot{\boldsymbol{u}}^T \bar{\boldsymbol{p}} + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}}\right) ds \\ &+ \sum_{H=A}^{B} \left[\dot{\boldsymbol{u}}^T \bar{\boldsymbol{P}} + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{C}}\right]_H - \int_{\mathcal{S}} \left(\lambda_2\dot{\gamma}_2 + \lambda_3\dot{\gamma}_3\right) ds \end{aligned} \qquad [4.13]$$

where λ_2, λ_3 are Lagrangian multipliers. Since this equation coincides with equation [4.9] when $\lambda_2 \equiv T_2$, $\lambda_3 \equiv T_3$ are taken, the Lagrangian balance equations [2.147] are obtained, namely:

$$\begin{aligned} &\boldsymbol{R}\left(\boldsymbol{t}' + \boldsymbol{K}\boldsymbol{t}\right) + \bar{\boldsymbol{p}} = \boldsymbol{0} \\ &\bar{\boldsymbol{B}}_\omega^T \left[\boldsymbol{R}\left(\boldsymbol{m}' + \boldsymbol{K}\boldsymbol{m}\right) + \boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{t} + \bar{\boldsymbol{c}}\right] = \boldsymbol{0} \end{aligned} \qquad [4.14]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

together with the boundary conditions [2.148]:

$$\begin{aligned}&\dot{\boldsymbol{u}}_H^T\left(\bar{\boldsymbol{P}}_H \pm \boldsymbol{R}_H\boldsymbol{t}_H\right)=\boldsymbol{0}\\&\dot{\boldsymbol{\theta}}_H^T\bar{\boldsymbol{B}}_{\omega H}^T\left(\bar{\boldsymbol{C}}_H \pm \boldsymbol{R}_H\boldsymbol{m}_H\right)=\boldsymbol{0}\end{aligned} \qquad [4.15]$$

On the other hand, the elastic law only concerns the active stresses (equation [1.94]). By adopting the diagonal linear elastic law for the unconstrained model (equations [2.164]), and appending dummy equations for the reactive stresses, we have:

$$\begin{aligned}\boldsymbol{t}&=\mathring{\boldsymbol{t}}+EA\varepsilon\bar{\boldsymbol{a}}_1+\boldsymbol{\lambda}\\\boldsymbol{m}&=\mathring{\boldsymbol{m}}+\boldsymbol{E}_{kk}\boldsymbol{k}\end{aligned} \qquad [4.16]$$

where:

$$\begin{aligned}&\mathring{\boldsymbol{t}}:=\begin{pmatrix}\mathring{N}\\0\\0\end{pmatrix}, \qquad \boldsymbol{\lambda}:=\begin{pmatrix}0\\T_2\\T_3\end{pmatrix}\\&\mathring{\boldsymbol{m}}:=\begin{pmatrix}\mathring{M}_1\\\mathring{M}_2\\\mathring{M}_3\end{pmatrix}, \quad \boldsymbol{E}_{kk}:=\begin{bmatrix}GJ_1&0&0\\0&GJ_2&0\\0&0&GJ_3\end{bmatrix}\end{aligned} \qquad [4.17]$$

In conclusion, the Fundamental Problem is made up of six scalar balance equations [4.14]; four unconstrained strain–displacement relationships, equation [4.11]; two equations expressing the vanishing of the shear strains, equations [4.12]; four (meaningful) scalar constitutive laws for the active stresses, equations [4.16]; overall, 16 equations. The unknowns are in the same number, namely: six displacement components $\boldsymbol{u}$, $\boldsymbol{\theta}$; four unconstrained strains ε, $\boldsymbol{k}$; four active stresses N, $\boldsymbol{t}$ and two reactive stresses T_2, T_3. Boundary conditions are mechanical (equations [4.15]) and/or geometrical (equations [4.4]).

In the spirit of the mixed formulation, by using the constitutive law and the strain-displacement relationships, the whole problem can be formulated as six balance equations with two constrain conditions appended, all expressed in terms of the six displacements and the two reactive stresses. In formulas, in the unprestressed case, and after premultiplying them by $\boldsymbol{R}^T$ for an easier expression, we have

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

(balance of momentum):

$$\begin{aligned}
&\{EA[(1+u_1')\cos\theta_2\cos\theta_3 + u_2'\cos\theta_2\sin\theta_3 - u_3'\sin\theta_2 - 1]\}' \\
&\quad - [\theta_3'\cos\theta_1\cos\theta_2 - \theta_2'\sin\theta_1]T_2 + [\theta_2'\cos\theta_1 + \theta_3'\sin\theta_1\cos\theta_2]T_3 \\
&\quad + \cos\theta_1\cos\theta_3(\bar{p}_1 - m\ddot{u}_1) \\
&\quad + \cos\theta_2\sin\theta_3(\bar{p}_2 - m\ddot{u}_2) - \sin\theta_2(\bar{p}_3 - m\ddot{u}_3) = 0 \\
&T_2' + EA[\theta_3'\cos\theta_1\cos\theta_2 - \theta_2'\sin\theta_1][(1+u_1')\cos\theta_2\cos\theta_3 \\
&\quad + u_2'\cos\theta_2\sin\theta_3 - u_3'\sin\theta_2 - 1] - [\theta_1' - \theta_3'\sin\theta_2]T_3 \\
&\quad + [\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3](\bar{p}_1 - m\ddot{u}_1) \\
&\quad + [\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3](\bar{p}_2 - m\ddot{u}_2) \\
&\quad + \sin\theta_1\cos\theta_2(\bar{p}_3 - m\ddot{u}_3) = 0 \\
&T_3' - EA[\theta_2'\cos\theta_1 + \theta_3'\sin\theta_1\cos\theta_2][(1+u_1')\cos\theta_2\cos\theta_3 \\
&\quad + u_2'\cos\theta_2\sin\theta_3 - u_3'\sin\theta_2 - 1] + [\theta_1' - \theta_3'\sin\theta_2]T_2 \\
&\quad + [\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3](\bar{p}_1 - m\ddot{u}_1) \\
&\quad + [\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3](\bar{p}_2 - m\ddot{u}_2) \\
&\quad + \cos\theta_1\cos\theta_2(\bar{p}_3 - m\ddot{u}_3) = 0
\end{aligned} \tag{4.18}$$

and (balance of angular momentum):

$$\begin{aligned}
&\{GJ_1[\theta_1' - \theta_3'\sin\theta_2]\}' + (EJ_3 - EJ_2)[\theta_2'\cos\theta_1 \\
&\quad + \theta_3'\sin\theta_1\cos\theta_2][\theta_3'\cos\theta_1\cos\theta_2 - \theta_2'\sin\theta_1] \\
&\quad + \cos\theta_1\cos\theta_3\bar{c}_1 + \cos\theta_2\sin\theta_3\bar{c}_2 - \sin\theta_2\bar{c}_3 = 0 \\
&\{EJ_2[\theta_2'\cos\theta_1 + \theta_3'\sin\theta_1\cos\theta_2]\}' \\
&\quad + (GJ_1 - EJ_3)[\theta_3'\cos\theta_1\cos\theta_2 - \theta_2'\sin\theta_1][\theta_1' - \theta_3'\sin\theta_2] \\
&\quad - T_3[(1+u_1')\cos\theta_2\cos\theta_3 + u_2'\cos\theta_2\sin\theta_3 - u_3'\sin\theta_2] \\
&\quad + [\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3]\bar{c}_1 \\
&\quad + [\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3]\bar{c}_2 + \sin\theta_1\cos\theta_2\bar{c}_3 = 0 \\
&\{EJ_3[\theta_3'\cos\theta_1\cos\theta_2 - \theta_2'\sin\theta_1]\}' \\
&\quad + (EJ_2 - GJ_1)[\theta_1' - \theta_3'\sin\theta_2][\theta_2'\cos\theta_1 + \theta_3'\sin\theta_1\cos\theta_2] \\
&\quad + T_2[(1+u_1')\cos\theta_2\cos\theta_3 + u_2'\cos\theta_2\sin\theta_3 - u_3'\sin\theta_2] \\
&\quad + [\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3]\bar{c}_1 \\
&\quad + [\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3]\bar{c}_2 + \cos\theta_1\cos\theta_2\bar{c}_3 = 0
\end{aligned} \tag{4.19}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with the boundary conditions:

$$
\begin{aligned}
&[\pm EA[(1+u_1')\cos\theta_2\cos\theta_3 + u_2'\cos\theta_2\sin\theta_3 - u_3'\sin\theta_2 - 1] \\
&\quad + \cos\theta_1\cos\theta_3\bar{P}_1 + \cos\theta_2\sin\theta_3\bar{P}_2 - \sin\theta_2\bar{P}_3]_H = 0 \\
&[\pm T_2 + [\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3]\bar{P}_1 \\
&\quad + [\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3]\bar{P}_2 + \sin\theta_1\cos\theta_2\bar{P}_3]_H = 0 \\
&[\pm T_3 + [\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3]\bar{P}_1 \\
&\quad + [\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3]\bar{P}_2 + \cos\theta_1\cos\theta_2\bar{P}_3]_H = 0 \\
&[\pm GJ_1[\theta_1' - \theta_3'\sin\theta_2] + \cos\theta_1\cos\theta_3\bar{C}_1 \\
&\quad + \cos\theta_2\sin\theta_3\bar{C}_2 - \sin\theta_2\bar{C}_3]_H = 0 \\
&[\pm EJ_2[\theta_2'\cos\theta_1 + \theta_3'\sin\theta_1\cos\theta_2] + [\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3]\bar{C}_1 \\
&\quad + [\cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3]\bar{C}_2 + \sin\theta_1\cos\theta_2\bar{C}_3]_H = 0 \\
&[\pm EJ_3[\theta_3'\cos\theta_1\cos\theta_2 - \theta_2'\sin\theta_1] + [\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3]\bar{C}_1 \\
&\quad + [\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3]\bar{C}_2 + \cos\theta_1\cos\theta_2\bar{C}_3]_H = 0
\end{aligned}
\qquad [4.20]
$$

The constraints equations [4.12] become:

$$
\begin{aligned}
&(1+u_1')(\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3) + u_2'(\cos\theta_1\cos\theta_3 \\
&\quad + \sin\theta_1\sin\theta_2\sin\theta_3) + u_3'\sin\theta_1\cos\theta_2 = 0 \\
&(1+u_1')(\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3) + u_2'(\cos\theta_1\sin\theta_2\sin\theta_3 \\
&\quad - \sin\theta_1\cos\theta_3) + u_3'\cos\theta_1\cos\theta_2 = 0
\end{aligned}
\qquad [4.21]
$$

4.3.2 The displacement formulation

The constraints

According to the displacement formulation, we must eliminate two (slave) variables, conveniently chosen as θ_2, θ_3, from the constrains [4.12], in order to express them as functions of the remaining four. Hence, $\boldsymbol{u}, \theta_1$ are master displacements and θ_2, θ_3 are slave displacements. However, solving equations [4.12] is not so easy. Therefore, we prefer to follow an equivalent, but simpler, method that better highlights the geometrical meaning of the constraints. Actually, we know from kinematics that the unit vector $\bar{\mathbf{a}}_1$, normal to the cross-section in the reference configuration, transforms into the unit vector $\mathbf{R}\bar{\mathbf{a}}_1$ in the current configuration. Moreover, the unit vector $\bar{\mathbf{x}}' = \bar{\mathbf{a}}_1$, tangent to the centerline transforms into the vector $\mathbf{x}' = \bar{\mathbf{a}}_1 + \mathbf{u}'$, still tangent to the centerline, but no more unitary; if no shear occurs, then $\|\mathbf{x}'\| = 1 + \varepsilon$, where $\varepsilon \equiv e$ is the unit strain. Shear-indeformability also

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

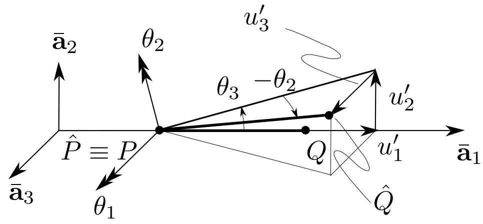


Figure 4.1: Geometrical interpretation of the unshearability ($\overrightarrow{PQ} = ds = 1$).

requires that cross-section and centerline remain mutually orthogonal in the current configuration, i.e.:

$$\bar{\mathbf{a}}_1 + \mathbf{u}' = (1 + \varepsilon)\, \mathbf{R}\bar{\mathbf{a}}_1 \qquad [4.22]$$

which expresses the parallelism between $\mathbf{x}'$ and $\mathbf{R}\bar{\mathbf{a}}_1$. When this vector equation is projected onto the $\bar{\mathcal{B}}$-basis, and equation [2.6] is used, it becomes:

$$\begin{pmatrix} 1 + u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = (1 + \varepsilon) \begin{pmatrix} \cos\theta_2 \cos\theta_3 \\ \cos\theta_2 \sin\theta_3 \\ -\sin\theta_2 \end{pmatrix} \qquad [4.23]$$

from which, by eliminating ε, the unknowns θ_2, θ_3 can easily be derived as:

$$\theta_2 = \arctan\left(-\frac{u'_3}{\sqrt{(1 + u'_1)^2 + u'^2_2}} \right), \qquad \theta_3 = \arctan\left(\frac{u'_2}{1 + u'_1} \right) \qquad [4.24]$$

which replace equations [4.12]. Relationships [4.24] are susceptible to the geometrical representation of Figure 4.1, in which the rotations of the centerline segment are identified with the rotations θ_2, θ_3 of the cross-section.

Strain–displacement relationships

From equation [4.23], the unit strain also follows so that, by appending equation [4.11-b], the unconstrained strains become:

$$\begin{aligned} \varepsilon &= \sqrt{(1 + u'_1)^2 + u'^2_2 + u'^2_3} - 1 \\ \boldsymbol{k} &= \bar{\boldsymbol{B}}_\omega \boldsymbol{\theta}' \end{aligned} \qquad [4.25]$$

In the expressions of the curvatures, however, $\theta_2 = \theta_2(\boldsymbol{u}')$, $\theta_3 = \theta_3(\boldsymbol{u}')$ must be understood, according to equation [4.24], so that all strains are expressed in terms of the master variables (see equation [1.45]).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Velocity constraints

As a consequence of the constraints [4.24], the velocity field is also constrained. Indeed, by time-differentiating these equations, the rates of the slave variables are expressed in terms of the rates of the master variables; by appending the dummy equation $\dot{\theta}_1 = \dot{\theta}_1$ (which expresses the fact that $\dot{\theta}_1$ is unconstrained), we have:

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix} \begin{pmatrix} \dot{u}'_1 \\ \dot{u}'_2 \\ \dot{u}'_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{\theta}_1 \qquad [4.26]$$

or, in compact form:

$$\dot{\boldsymbol{\theta}} = \boldsymbol{A}_\theta \dot{\boldsymbol{u}}' + \bar{\boldsymbol{a}}_1 \dot{\theta}_1 \qquad [4.27]$$

Here, $\boldsymbol{A}_\theta := [a_{ij}]$ has the following non-zero entries[5]:

$$\begin{aligned} a_{21} &:= \frac{(1+u'_1)u'_3}{\left((1+u'_1)^2 + u'^2_2\right)^{1/2} \left((1+u'_1)^2 + u'^2_2 + u'^2_3\right)} \\ a_{22} &:= \frac{u'_2 u'_3}{\left((1+u'_1)^2 + u'^2_2\right)^{1/2} \left((1+u'_1)^2 + u'^2_2 + u'^2_3\right)} \\ a_{23} &:= -\frac{(1+u'_1)^2 + u^2_2}{\left((1+u'_1)^2 + u'^2_2\right)^{1/2} \left((1+u'_1)^2 + u'^2_2 + u'^2_3\right)} \\ a_{31} &:= \frac{u'_2}{(1+u'_1)^2 + u'^2_2}, \qquad a_{32} = \frac{1+u'_1}{(1+u'_1)^2 + u'^2_2} \end{aligned} \qquad [4.28]$$

Strain-rates

Admissible strain-rates are the time-derivatives of the strains [4.25]; $\dot{\boldsymbol{k}}$, however, is more conveniently expressed by equation [4.5b]. Therefore:

$$\begin{aligned} \dot{\varepsilon} &= \boldsymbol{d}_\varepsilon^T \dot{\boldsymbol{u}}' \\ \dot{\boldsymbol{k}} &= \boldsymbol{R}^T \left(\bar{\boldsymbol{B}}'_\omega \dot{\boldsymbol{\theta}} + \bar{\boldsymbol{B}}_\omega \dot{\boldsymbol{\theta}}' \right) \end{aligned} \qquad [4.29]$$

5. The matrix $\boldsymbol{A}_\theta$ and the column matrix $\bar{\boldsymbol{a}}_1$ are pieces of the matrix $\boldsymbol{A}$ introduced for the metamodel (equation [1.43]), relating velocities and master velocities.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where $\boldsymbol{d}_\varepsilon = \partial\varepsilon/\partial\boldsymbol{u}' := (d_1, d_2, d_3)^T$ is a column matrix, whose components are[6]:

$$
\begin{aligned}
d_1 &:= \frac{1+u_1'}{\sqrt{(1+u_1')^2+u_2'^2+u_3'^2}}, \quad d_2 := \frac{u_2'}{\sqrt{(1+u_1')^2+u_2'^2+u_3'^2}} \\
d_3 &:= \frac{u_3'}{\sqrt{(1+u_1')^2+u_2'^2+u_3'^2}}
\end{aligned}
\qquad [4.30]
$$

The balance equations

Balance equations are derived by the VPP [4.9], in which $\dot\gamma_2 = 0$, $\dot\gamma_3 = 0$, i.e.:

$$
\begin{aligned}
&\int_{\mathcal{S}} \left(N\dot\varepsilon + \dot{\boldsymbol{k}}^T\boldsymbol{m}\right) ds \\
&= \int_{\mathcal{S}} \left(\dot{\boldsymbol{u}}^T\bar{\boldsymbol{p}} + \dot{\boldsymbol{\theta}}^T\bar{\boldsymbol{B}}_\omega^T\bar{\boldsymbol{c}}\right) ds + \sum_{H=A}^{B}\left[\dot{\boldsymbol{u}}^T\bar{\boldsymbol{P}} + \dot{\boldsymbol{\theta}}^T\bar{\boldsymbol{B}}_\omega^T\bar{\boldsymbol{C}}\right]_H
\end{aligned}
\qquad [4.31]
$$

According to the general strategy we have outlined, we first substitute the strain rates [4.29] in the VPP, by ignoring the constraints on the velocities, and perform an integration by parts. Since the internal power is:

$$
\begin{aligned}
\mathcal{P}_{int} &:= \int_{\mathcal{S}} \left[\dot{\boldsymbol{u}}'^T(N\boldsymbol{d}_\varepsilon) + \left(\dot{\boldsymbol{\theta}}^T\bar{\boldsymbol{B}}_\omega'^T + \dot{\boldsymbol{\theta}}'^T\bar{\boldsymbol{B}}_\omega^T\right)(\boldsymbol{R}\boldsymbol{m})\right] ds \\
&= \int_{\mathcal{S}} \left[-\dot{\boldsymbol{u}}^T(N\boldsymbol{d}_\varepsilon)' - \dot{\boldsymbol{\theta}}^T\left(\bar{\boldsymbol{B}}_\omega^T\boldsymbol{R}\boldsymbol{m}\right)' + \dot{\boldsymbol{\theta}}^T\left(\bar{\boldsymbol{B}}_\omega'^T\boldsymbol{R}\boldsymbol{m}\right)\right] ds \\
&\quad + \left[\dot{\boldsymbol{u}}^T(N\boldsymbol{d}_\varepsilon) + \dot{\boldsymbol{\theta}}^T\left(\bar{\boldsymbol{B}}_\omega^T\boldsymbol{R}\boldsymbol{m}\right)\right]_A^B
\end{aligned}
\qquad [4.32]
$$

the VPP becomes:

$$
\begin{aligned}
&\int_{\mathcal{S}} \left\{\dot{\boldsymbol{u}}^T\left[(N\boldsymbol{d}_\varepsilon)' + \bar{\boldsymbol{p}}\right] + \dot{\boldsymbol{\theta}}^T\left[\left(\bar{\boldsymbol{B}}_\omega^T\boldsymbol{R}\boldsymbol{m}\right)' + \bar{\boldsymbol{B}}_\omega^T\bar{\boldsymbol{c}} - \bar{\boldsymbol{B}}_\omega'^T\boldsymbol{R}\boldsymbol{m}\right]\right\} ds \\
&+ \sum_{H=A}^{B}\left[\dot{\boldsymbol{u}}^T\left(\bar{\boldsymbol{P}} \pm N\boldsymbol{d}_\varepsilon\right) + \dot{\boldsymbol{\theta}}^T\bar{\boldsymbol{B}}_\omega^T\left(\bar{\boldsymbol{C}} \pm \boldsymbol{R}\boldsymbol{m}\right)\right]_H = 0
\end{aligned}
\qquad [4.33]
$$

6. Note that $\boldsymbol{d}_\varepsilon$ is a piece of the kinematic operator $\boldsymbol{D}$; the symbol should *not* be confused with $\mathbf{d}$, which is the stretching velocity vector.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

In these equations, however, $\dot{\boldsymbol{\theta}}$ is *not* a free variable, since it is restrained by equation [4.27]. Therefore, by substituting it under the integral sign (only) and integrating a second time by parts, the VPP becomes:[7]

$$\begin{aligned}
&\int_{\mathcal{S}} \left\{\dot{\boldsymbol{u}}^T \left[(N\boldsymbol{d}_\varepsilon)' + \bar{\boldsymbol{p}}\right]\right\} ds \\
&\quad + \sum_{H=A}^{B} \left[\dot{\boldsymbol{u}}^T \left(\bar{\boldsymbol{P}} \pm N\boldsymbol{d}_\varepsilon\right) + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \left(\bar{\boldsymbol{C}} \pm \boldsymbol{Rm}\right)\right]_H \\
&\quad + \int_{\mathcal{S}} \left\{-\dot{\boldsymbol{u}}^T \left[\boldsymbol{A}_\theta^T \left(\left(\bar{\boldsymbol{B}}_\omega^T \boldsymbol{Rm}\right)' + \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}} - \bar{\boldsymbol{B}}_\omega'^T \boldsymbol{Rm}\right)\right]'\right\} ds \\
&\quad + \int_{\mathcal{S}} \left\{\dot{\theta}_1 \bar{\boldsymbol{a}}_1^T \left[\left(\bar{\boldsymbol{B}}_\omega^T \boldsymbol{Rm}\right)' + \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}} - \bar{\boldsymbol{B}}_\omega'^T \boldsymbol{Rm}\right]\right\} ds \\
&\quad + \left[\dot{\boldsymbol{u}}^T \left[\boldsymbol{A}_\theta^T \left(\left(\bar{\boldsymbol{B}}_\omega^T \boldsymbol{Rm}\right)' + \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}} - \bar{\boldsymbol{B}}_\omega'^T \boldsymbol{Rm}\right)\right]\right]_A^B = 0
\end{aligned} \qquad [4.34]$$

By requiring that the power balance holds for any $\dot{\boldsymbol{u}}$, $\dot{\theta}_1$, the following four (condensed) local balance equations are derived (see equation [1.54]):

$$\begin{aligned}
&\left[N\boldsymbol{d}_\varepsilon - \boldsymbol{A}_\theta^T \left(\left(\bar{\boldsymbol{B}}_\omega^T \boldsymbol{Rm}\right)' - \bar{\boldsymbol{B}}_\omega'^T \boldsymbol{Rm} + \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}}\right)\right]' + \bar{\boldsymbol{p}} = \boldsymbol{0} \\
&\bar{\boldsymbol{a}}_1^T \left[\left(\bar{\boldsymbol{B}}_\omega^T \boldsymbol{Rm}\right)' - \bar{\boldsymbol{B}}_\omega'^T \boldsymbol{Rm} + \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}}\right] = 0
\end{aligned} \qquad [4.35]$$

Moreover, by collecting and vanishing the coefficients of $\dot{\boldsymbol{u}}$, $\dot{\boldsymbol{\theta}}_1$ at the beam ends, the following alternative boundary conditions are obtained (see equation [1.55]):

$$\begin{aligned}
&\dot{\boldsymbol{u}}_H^T \left[\left(\bar{\boldsymbol{P}} \pm N\boldsymbol{d}_\varepsilon\right) \mp \boldsymbol{A}_\theta^T \left(\left(\bar{\boldsymbol{B}}_\omega^T \boldsymbol{Rm}\right)' + \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}} - \bar{\boldsymbol{B}}_\omega'^T \boldsymbol{Rm}\right)\right]_H = 0 \\
&\dot{\boldsymbol{\theta}}_H^T \left[\bar{\boldsymbol{B}}_\omega^T \left(\bar{\boldsymbol{C}} \pm \boldsymbol{Rm}\right)\right]_H = 0
\end{aligned} \qquad [4.36]$$

7. Note that substitution of the constraint [4.27] at the boundary is useless, since the mechanical devices which are applied there decide if $\dot{\boldsymbol{\theta}}_H$ is admissible or not, regardless of the fact that the spin is expressed in terms of master variables (as we did in equation [1.50] or not (as we are doing in equation [4.34]).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The elastic law

The elastic law for the active stresses is as in equation [4.16], but without reactive stresses, namely:

$$\begin{aligned} N &= \mathring{N} + EA\varepsilon \\ \boldsymbol{m} &= \mathring{\boldsymbol{m}} + \boldsymbol{E}_{kk}\boldsymbol{k} \end{aligned} \qquad [4.37]$$

The Fundamental Problem

According to the displacement method, the condensed balance equations [4.35] and [4.36] must be expressed in terms of master variables. By using the constitutive law [4.37] and the condensed strain–displacement relationships [4.25], in which the constraints [4.24] have been accounted for, four equations of motion in $\boldsymbol{u}, \theta_1$ are finally obtained.

4.4 The unshearable straight planar beam

The unshearable beam is now studied in the planar case in the framework of the displacement formulation only. Although the relevant equations could be obtained as a particularization of the 3D-problem, we find it more instructive to restart the procedure to corroborate its understanding in a more manageable case.

Kinematics

Strains of the planar beam were obtained in equations [2.206] as:

$$\begin{aligned} \varepsilon &= -1 + (1 + u_1') \cos\theta + u_2' \sin\theta \\ \gamma &= -(1 + u_1') \sin\theta + u_2' \cos\theta \\ \kappa &= \theta' \end{aligned} \qquad [4.38]$$

Unshearability requires that $\gamma = 0$, from which the slave variable θ is obtained in terms of the master variables u_1, u_2:

$$\theta = \arctan \frac{u_2'}{1 + u_1'} \qquad [4.39]$$

or, equivalently:

$$\cos\theta = \frac{1 + u_1'}{\sqrt{(1 + u_1')^2 + u_2'^2}}, \qquad \sin\theta = \frac{u_2'}{\sqrt{(1 + u_1')^2 + u_2'^2}} \qquad [4.40]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

When these results are used, the unconstrained strains become:

$$\begin{aligned}\varepsilon &= \sqrt{(1+u_1')^2 + u_2'^2} - 1 \\ \kappa &= \frac{(1+u_1')u_2'' - u_1''u_2'^2}{(1+u_1')^2 + u_2'^2}\end{aligned} \qquad [4.41]$$

Time-differentiation of equation [4.39] yields the velocity constraint:

$$\dot{\theta} = a_1 \dot{u}_1' + a_2 \dot{u}_2' \qquad [4.42]$$

where:

$$a_1 := -\frac{u_2'}{(1+u_1')^2 + u_2'^2}, \qquad a_2 := \frac{1+u_1'}{(1+u_1')^2 + u_2'^2} \qquad [4.43]$$

Time-differentiation of the unconstrained strains [4.41] provides the strain-rates:

$$\begin{aligned}\dot{\varepsilon} &= \dot{u}_1' \cos\theta + \dot{u}_2' \sin\theta \\ \dot{\kappa} &= \dot{\theta}'\end{aligned} \qquad [4.44]$$

In this latter equation, $\dot{\theta}$ must be understood as given by equation [4.42].

Balance equations

The virtual power balance [2.215], with $\dot{\gamma} = 0$, becomes:

$$\begin{aligned}\int_{\mathcal{S}} (N\dot{\varepsilon} + M\dot{\kappa})\, ds &= \int_{\mathcal{S}} \left(\bar{p}_1 \dot{u}_1 + \bar{p}_2 \dot{u}_2 + \bar{c}\dot{\theta}\right) ds \\ &+ \sum_{H=A}^{B} \left[\bar{P}_1 \dot{u}_1 + \bar{P}_2 \dot{u}_2 + \bar{C}\dot{\theta}\right]_H\end{aligned} \qquad [4.45]$$

By substituting equations [4.44] for $\dot{\varepsilon}, \dot{\kappa}$, and integrating a first time by parts, we obtain:

$$\begin{aligned}&\int_{\mathcal{S}} \left[-(N\cos\theta)'\dot{u}_1 - (N\sin\theta)'\dot{u}_2 - M'\dot{\theta}\right] ds \\ &+ \left[N\left(\dot{u}_1 \cos\theta + \dot{u}_2 \sin\theta\right) + M\dot{\theta}\right]_A^B = \int_{\mathcal{S}} \left(\bar{p}_1 \dot{u}_1 + \bar{p}_2 \dot{u}_2 + \bar{c}\dot{\theta}\right) ds \\ &+ \sum_{H=A}^{B} \left[\bar{P}_1 \dot{u}_1 + \bar{P}_2 \dot{u}_2 + \bar{C}\dot{\theta}\right]_H\end{aligned} \qquad [4.46]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Then, by substituting the velocity constraint [4.42] under the integral signs and performing a second integration by parts, we have:

$$\begin{aligned}
&\int_{\mathcal{S}} [(-N\cos\theta + M'a_1)'\dot{u}_1 + (-N\sin\theta + M'a_2)'\dot{u}_2]\, ds \\
&\quad + \left[(N\cos\theta - M'a_1)\,\dot{u}_1 + (N\sin\theta - M'a_2)\,\dot{u}_2 + M\dot{\theta}\right]_A^B \\
&\quad = \int_{\mathcal{S}} [(\bar{p}_1 - (\bar{c}a_1)')\dot{u}_1 + (\bar{p}_2 - (\bar{c}a_2)')\dot{u}_2)]\, ds \\
&\quad + \sum_{H=A}^{B} \left[\bar{P}_1\dot{u}_1 + \bar{P}_2\dot{u}_2 + \bar{C}\dot{\theta}\right]_H + [\bar{c}a_1\dot{u}_1 + \bar{c}a_2\dot{u}_2]_A^B
\end{aligned} \qquad [4.47]$$

Finally, by equating to zero the coefficients of $\dot{u}_1, \dot{u}_2$ in the field, and the coefficients of $\dot{u}_1, \dot{u}_2, \dot{\theta}$ at the boundary, the following balance equations are obtained:

$$\begin{aligned}
&(N\cos\theta)' - [(M' + \bar{c})a_1]' + \bar{p}_1 = 0 \\
&(N\sin\theta)' - [(M' + \bar{c})a_2]' + \bar{p}_2 = 0
\end{aligned} \qquad [4.48]$$

with the alternative boundary conditions:

$$\begin{aligned}
&\dot{u}_{1H}\left[\bar{P}_1 \pm (N\cos\theta - (M' + \bar{c})a_1)\right]_H = 0 \\
&\dot{u}_{2H}\left[\bar{P}_2 \pm (N\sin\theta - (M' + \bar{c})a_2)\right]_H = 0 \\
&\dot{\theta}_H\left[\bar{C} \pm M\right]_H = 0
\end{aligned} \qquad [4.49]$$

The Fundamental Problem

Once the uncoupled linear elastic law is used, and the unconstrained strains are expressed in terms of the master displacements via equations [4.41], the Fundamental Problem is described by the following equations, when the d'Alembert principle is used to introduce the inertia forces:

$$\begin{aligned}
&\left[EA\varepsilon\frac{(1+u_1')}{1+\varepsilon}\right]' + \left[\left(\left(EJ\frac{(1+u_1')\,u_2'' - u_1''u_2'}{(1+\varepsilon)^2}\right)' + \bar{c}\right)\frac{u_2'}{(1+\varepsilon)^2}\right]' \\
&+ \bar{p}_1 - m\ddot{u}_1 = 0 \\
&\left[EA\varepsilon\frac{u_2'}{1+\varepsilon}\right]' - \left[\left(\left(EJ\frac{(1+u_1')\,u_2'' - u_1''u_2'}{(1+\varepsilon)^2}\right)' + \bar{c}\right)\frac{1+u_1'}{(1+\varepsilon)^2}\right]' \\
&+ \bar{p}_2 - m\ddot{u}_2 = 0
\end{aligned} \qquad [4.50]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

in which the strain ε, as given by equation [4.41a], has been used as a position to simplify the writing. Moreover, with the same criterion, the relevant mechanical boundary conditions are:

$$\begin{aligned}
&\mp\left[EA\varepsilon\frac{1+u_1'}{1+\varepsilon}\right.\\
&\qquad\left.+\left(\left(EJ\frac{(1+u_1')\,u_2''-u_1''u_2'}{(1+\varepsilon)^2}\right)'+\bar{c}\right)\frac{u_2'}{(1+\varepsilon)^2}\right]_H=\bar{P}_{1H}\\
&\mp\left[EA\varepsilon\frac{u_2'}{1+\varepsilon}\right.\\
&\qquad\left.-\left(\left(EJ\frac{(1+u_1')\,u_2''-u_1''u_2'}{(1+\varepsilon)^2}\right)'+\bar{c}\right)\frac{1+u_1'}{(1+\varepsilon)^2}\right]_H=\bar{P}_{2H}\\
&\mp\left[EJ\frac{(1+u_1')\,u_2''-u_1''u_2'}{(1+\varepsilon)^2}\right]_H=\bar{C}
\end{aligned}\qquad [4.51]$$

The geometric boundary conditions are:

$$u_{1H}=\breve{u}_{1H},\quad u_{2H}=\breve{u}_{2H},\quad \arctan\left(\frac{u_2'}{1+u_1'}\right)_H=\breve{\theta}_H \qquad [4.52]$$

REMARK 4.1. The exact equations [4.50] and [4.51] strongly simplify if we admit that $\varepsilon \ll 1$, so that $1+\varepsilon \simeq 1$. This approximation does not entail that the beam is inextensible, since it is still $\varepsilon \neq 0$ in the terms proportional to EA. With the same order of approximation, it could be assumed that $1+u_1' \simeq 1$, and the equations further simplified.

4.5 The inextensible and unshearable straight beam in 3D

We now introduce the model of an internally constrained beam, in which not only the shear strains, but even the extension is prevented. This beam is also known as (inextensible) *Euler's elastica*. We present a *hybrid approach* that combines the displacement and the mixed formulations for internally constrained beams. In this framework, we illustrate two different methodologies, in which the constraints are handled as they naturally appear (Version I), or they are suitably combined (Version II).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

4.5.1 Hybrid formulation: Version I

The constraints require that $\varepsilon = \gamma_2 = \gamma_3 = 0$ [8]. The unshearability conditions, as already found, allow elimination of two rotations (equations [4.24], here repeated):

$$\theta_2 = \arctan\left(-\frac{u_3'}{\sqrt{(1+u_1')^2 + u_2'^2}}\right), \qquad \theta_3 = \arctan\left(\frac{u_2'}{1+u_1'}\right) \qquad [4.53]$$

while the inextensibility condition, $\varepsilon = 0$, leads to:

$$(1+u_1')^2 + u_2'^2 + u_3'^2 - 1 = 0 \qquad [4.54]$$

We will account for the three constraints *as they are*, i.e. without any further manipulation, and refer to this approach as Version I.

It is apparent that the inextensibility condition, differently from the unshearability, does not permit any elimination of a variable via algebraic operations, since all of them are differentiated. For this reason, we will proceed to a *partial condensation*, i.e. (a) we will *eliminate* the two rotations, as requested by the displacement formulation, by assuming $\boldsymbol{u}$, θ_1 as "master displacements", while, (b) we will *append* the inextensibility constraint, as requested by the mixed formulation. For these reasons, we will call this method the *hybrid formulation*. In this, the locution "master displacements" should be accepted in a broader sense, as variables that identically satisfied *a part* of the constraint conditions not explicitly appended.

With these ideas in mind, we write the virtual power by taking $\dot{\boldsymbol{k}}$ as unique admissible strain-rate, given by equation [4.29], so that the virtual power density becomes $\dot{\boldsymbol{k}}^T\boldsymbol{m}$. However, the strain-rates depend on the displacement-rates $\dot{\boldsymbol{u}}$, $\dot{\theta}_1$ that are not free, since they have to satisfy the constraint $\dot{\varepsilon} = 0$. In order to account for the latter, we use the Lagrange multiplier technique, by adding the zero term $\lambda\dot{\varepsilon}$ to the virtual power, and we deal with $\dot{\boldsymbol{u}}$, $\dot{\theta}_1$ as if they were free variables. Thus, the VPP [4.9] becomes (compare it with equation [1.26]):

$$\begin{aligned} \int_{\mathcal{S}} \dot{\boldsymbol{k}}^T \boldsymbol{m} ds &= \int_{\mathcal{S}} \left(\dot{\boldsymbol{u}}^T \bar{\boldsymbol{p}} + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}}\right) ds \\ &+ \sum_{H=A}^{B} \left[\dot{\boldsymbol{u}}^T \bar{\boldsymbol{P}} + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{C}}\right]_H - \int_{\mathcal{S}} \lambda \dot{\varepsilon} ds \end{aligned} \qquad [4.55]$$

8. By following the reasoning of the previous section, we can say that not only must $\mathbf{x}'$ and $\mathbf{R}\bar{\mathbf{a}}_1$ be parallel, as in equation [4.22], but that $\|\mathbf{x}'\| = 1$. Therefore, the constraints require that $\mathbf{x}' = \mathbf{R}\bar{\mathbf{a}}_1$, or equivalently $\mathbf{e} = \mathbf{0}$ (see equation [2.24]).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

However, this equation coincides with that holding for the extensible case (equation [4.31]), if we rename the Lagrange multiplier λ as the axial force N. As a major result, *the balance equations turn out to be identical to those of the extensible case* (equations [4.35] and [4.36]). However, since N is a reactive stress, it does not appear in the elastic law that consequently reduces to equation [4.16b] (compare it with equation [1.30a]):

$$\boldsymbol{m} = \mathring{\boldsymbol{m}} + \boldsymbol{E}_{kk}\boldsymbol{k} \tag{4.56}$$

Overall, the Fundamental Problem is similar to that for the extensible beam, but with a constitutive equation replaced by the constraint equation [4.54]. Moreover, this constraint strongly simplifies the expression of the elements of the matrices $\boldsymbol{A}_\theta$ [4.28] and $\boldsymbol{d}_\varepsilon$ [4.30].

4.5.2 Hybrid formulation: Version II

There is an alternative way to handle constraints, which we want to discuss now. It consists of combining them in order to express the rotations θ_2, θ_3 in terms of the transverse displacements u_2, u_3 rather than of the longitudinal displacement u_1, the former being expected to be more relevant[9]. We will refer to this method as Version II.

Substitution of equation [4.54] in [4.53] leads to[10]:

$$\tan\theta_2 = -\frac{u_3'}{\sqrt{1-u_3'^2}}, \qquad \tan\theta_3 = \frac{u_2'}{\sqrt{1-u_2'^2-u_3'^2}} \tag{4.57}$$

By time-differentiating them, and appending a dummy equation, we have:

$$\begin{pmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{pmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & a_{23}\\ 0 & a_{32} & a_{33}\end{bmatrix}\begin{pmatrix}\dot u_1'\\ \dot u_2'\\ \dot u_3'\end{pmatrix} + \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\dot\theta_1 \tag{4.58}$$

which is still of the form [4.27], i.e.:

$$\dot{\boldsymbol{\theta}} = \boldsymbol{A}_\theta \dot{\boldsymbol{u}}' + \bar{\boldsymbol{a}}_1\dot\theta_1 \tag{4.59}$$

but where the coefficients of the matrix $\boldsymbol{A}_\theta$ are much simpler, namely:

$$\begin{aligned} a_{23} &:= -\frac{1}{\sqrt{1-u_3'^2}} \\ a_{32} &:= \frac{1}{\sqrt{1-u_2'^2-u_3'^2}}, \quad a_{33} := \frac{u_2' u_3'}{(1-u_3'^2)\sqrt{1-u_2'^2-u_3'^2}} \end{aligned} \tag{4.60}$$

9. The longitudinal displacement, indeed, is a second-order variable with respect to the transverse displacement, as a series expansion of equation [4.54] soon reveals.

10. These new expressions show that θ_2, θ_3 are *odd* functions of the transverse displacements.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

From now on, the procedure is identical to that of Version I, and therefore leads to the same balance equations [4.35] and [4.36]. However, due to the simpler form of $\boldsymbol{A}_\theta$, they contain a lower number of scalar terms. As a matter of fact, since $\boldsymbol{A}_\theta^T$ has zero entries in the first row, the first balance equations (expressing the balance of the forces along $\bar{\mathbf{a}}_1$) reduce to:

$$(\lambda d_1)' + \bar{p}_1 = 0 \qquad [4.61]$$

in which we preferred not to change the name of the Lagrangian multiplier. Indeed, since this equation differs from the previous version, we have to conclude that λ *assumes a different meaning than that of Version I*. We will investigate this circumstance in the subsequent section, by dealing with the simpler planar case. Formulation of the Fundamental Problem is, for the rest, identical to Version I.

REMARK 4.2. The circumstance that the expression of the balance equations depend on the way in which constraints are (*in equivalent ways*) enforced, should not be surprising! As a matter of fact, if, in rigid-body mechanics, a point is fixed by three non-planar and non-parallel pendula, the reactions of each pendulum depend on how these are oriented, although the resultant force is independent of the arrangement.

4.6 The inextensible and unshearable straight planar beam

We now explain how to formulate the planar model of Euler's elastica starting from the unconstrained beam (derivation from the more general spatial case of constrained beam is, of course, trivial). The two versions of the hybrid formulation, previously introduced, are discussed; moreover, a third approach, carried out in the spirit of the mixed formulation, is presented. The different approaches are useful to understand the differences encountered in handling constraints in different (but equivalent) manners.

4.6.1 Hybrid formulation: Version I

Unshearability of the planar beam calls for $\gamma = 0$; inextensibility for $\varepsilon = 0$. When the strains [4.38] are used, the following geometrical constraints are obtained:

$$\begin{aligned} \theta &= \arctan \frac{u_2'}{1+u_1'} \\ \varepsilon &= \sqrt{(1+u_1')^2 + u_2'^2} - 1 = 0 \end{aligned} \qquad [4.62]$$

From (both of) which, it follows that:

$$\cos\theta = 1 + u_1', \qquad \sin\theta = u_2' \qquad [4.63]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The only admissible strain is therefore $\kappa = \theta'$ (still given by equation [4.41]). Time differentiation of the constraints yields:

$$\begin{aligned}\dot{\theta} &= -u_2'\dot{u}_1' + (1+u_1')\,\dot{u}_2' \\ \dot{\varepsilon} &= (1+u_1')\,\dot{u}_1' + u_2'\dot{u}_2'\end{aligned} \qquad [4.64]$$

where $\varepsilon = 0$ has been taken into account. Moreover, $\dot{\kappa} = \dot{\theta}'$ and $\omega = \dot{\theta}$, with $\dot{\theta}$ given by the former equations.

The VPP, in which the inextensibility is enforced via a Lagrange multiplier λ, becomes:

$$\begin{aligned}\int_{\mathcal{S}} M\dot{\kappa}ds &= \int_{\mathcal{S}} \left(\bar{p}_1\dot{u}_1 + \bar{p}_2\dot{u}_2 + \bar{c}\dot{\theta}\right) ds \\ &\quad + \sum_{H=A}^{B} \left[\bar{P}_1\dot{u}_1 + \bar{P}_2\dot{u}_2 + \bar{C}\dot{\theta}\right]_H - \int_{\mathcal{S}} \lambda\dot{\varepsilon}ds\end{aligned} \qquad [4.65]$$

which is identical to equation [4.45], with $\lambda \equiv N$. Therefore, the same balance equations [4.48] and [4.49] are obtained, with a more simple expression of sine and cosine of the rotation angle. In these equations, moreover, N is a reactive force. Therefore, by expressing $M = EJ\kappa$, and using [4.41b] for κ, the Fundamental Problem is presented in the following form, when the d'Alembert principle is used to express the inertia forces:

$$\begin{aligned}&[N(1+u_1')]' + \left[\left((EJ((1+u_1')\,u_2'' - u_1''u_2'))' + \bar{c}\right) u_2'\right]' \\ &+ \bar{p}_1 - m\ddot{u}_1 = 0 \\ &(Nu_2')' - \left[\left((EJ((1+u_1')\,u_2'' - u_1''u_2'))' + \bar{c}\right)(1+u_1')\right]' \\ &+ \bar{p}_2 - m\ddot{u}_2 = 0\end{aligned} \qquad [4.66]$$

with the mechanical boundary conditions:

$$\begin{aligned}&\mp\left[N(1+u_1') + \left((EJ((1+u_1')\,u_2'' - u_1''u_2'))' + \bar{c}\right) u_2'\right]_H = \bar{P}_1 \\ &\mp\left[Nu_2' + \left((EJ((1+u_1')\,u_2'' - u_1''u_2'))' + \bar{c}\right)(1+u_1')\right]_H = \bar{P}_2 \\ &\mp\left[EJ((1+u_1')\,u_2'' - u_1''u_2')\right]_H = \bar{C}, \qquad H = A, B\end{aligned} \qquad [4.67]$$

and the geometric boundary conditions given by equation [4.52]. Moreover, the constraint [4.62b] must be appended.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

4.6.2 Hybrid formulation: Version II

If the constraints [4.62] are combined, the longitudinal displacement can be eliminated from the expression of the rotation to obtain:

$$\theta = \arctan \frac{u_2'}{\sqrt{1-u_2'^2}}$$
$$\varepsilon = \sqrt{(1+u_1')^2 + u_2'^2} - 1 = 0 \qquad [4.68]$$

Thus:

$$\cos\theta = \sqrt{1-u_2'^2}, \qquad \sin\theta = u_2' \qquad [4.69]$$

while:

$$\kappa = \frac{u_2''}{\sqrt{1-u_2'^2}} \qquad [4.70]$$

Time-differentiation of the modified constraint furnishes:

$$\dot\theta = \frac{\dot u_2'}{\sqrt{1-u_2'^2}} \equiv \frac{\dot u_2'}{\cos\theta} \qquad [4.71]$$

where $\varepsilon = 0$ has been taken into account. Moreover, $\dot\kappa = \dot\theta'$ and $\omega = \dot\theta$, with $\dot\theta$ given by the former equations.

The VPP, with the Lagrange multiplier λ, is still given by equation [4.65], and the first integration by parts still leads to equation [4.46], with λ instead of N. From now on, however, the procedure differs, since we have to account for the new expression of $\dot\theta$, leading to:

$$\int_{\mathcal{S}} \left[-(\lambda\cos\theta)'\dot u_1 - (\lambda\sin\theta)'\dot u_2 - M' \frac{\dot u_2'}{\cos\theta} \right] ds$$
$$+ \left[\lambda\left(\dot u_1\cos\theta + \dot u_2 \sin\theta\right) + M \frac{\dot u_2'}{\cos\theta} \right]_A^B \qquad [4.72]$$
$$= \int_{\mathcal{S}} \left(\bar p_1 \dot u_1 + \bar p_2 \dot u_2 + \bar c \frac{\dot u_2'}{\cos\theta} \right) ds + \sum_{H=A}^{B} \left[\bar P_1 \dot u_1 + \bar P_2 \dot u_2 + \bar C \frac{\dot u_2'}{\cos\theta} \right]_H$$

When a second integration by parts is performed to free u_2' from the space-derivative, the following balance equations are obtained:

$$(\lambda\cos\theta)' + \bar p_1 = 0$$
$$(\lambda\sin\theta)' - \left(\frac{M' + \bar c}{\cos\theta}\right)' + \bar p_2 = 0 \qquad [4.73]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with the mechanical boundary conditions:

$$\begin{aligned} &\left[\bar{P}_1 \pm \lambda \cos\theta\right]_H \dot{u}_{1H} = 0 \\ &\left[\bar{P}_2 \pm \left(-\lambda \sin\theta - \frac{M' + \bar{c}}{\cos\theta}\right)\right]_H \dot{u}_{2H} = 0 \\ &\left[\bar{C} \pm M\right]_H \dot{\theta}_H = 0 \end{aligned} \qquad [4.74]$$

Once the bending moment is expressed via the elastic law and the d'Alembert principle is used, the Fundamental Problem is as follows:

$$\begin{aligned} &\left[\lambda\sqrt{1 - u_2'^2}\right]' + \bar{p}_1 - m\ddot{u}_1 = 0 \\ &(\lambda u_2')' - \left(\frac{1}{\sqrt{1 - u_2'^2}}\left[\left(EJ\frac{u_2''}{\sqrt{1 - u_2'^2}}\right)' + \bar{c}\right]\right)' + \bar{p}_2 - m\ddot{u}_2 = 0 \end{aligned} \qquad [4.75]$$

and, on the boundary:

$$\begin{aligned} &\mp\left[\lambda\sqrt{1 - u_2'^2}\right]_H = \bar{P}_{1H} \\ &\mp\left[\lambda u_2' - \frac{1}{\sqrt{1 - u_2'^2}}\left[\left(EJ\frac{u_2''}{\sqrt{1 - u_2'^2}}\right)' + \bar{c}\right]\right]_H = \bar{P}_{2H} \\ &\mp\left[EJ\frac{u_2''}{\sqrt{1 - u_2'^2}}\right]_H = \bar{C}_H \end{aligned} \qquad [4.76]$$

together with:

$$u_{1H} = \breve{u}_{1H}, \quad u_{2H} = \breve{u}_{2H}, \quad \left[\arctan\frac{u_2'}{\sqrt{1 - u_2'^2}}\right]_H = \breve{\theta}_{3H} \qquad [4.77]$$

Moreover, the inextensibility condition [4.68b] must be appended.

4.6.3 The mixed formulation

So far, we have been following a hybrid approach, in which the rotation θ is condensed via the unshearability condition, while inextensibility is accounted for by a Lagrange multiplier. Here, we want to follow a different approach in which *all* the constraints are dealt with as auxiliary conditions.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

First, we note that the pair of constraints, $\varepsilon = 0$, $\gamma = 0$, can be solved to furnish the displacements u_1, u_2 in terms of the rotation θ (equations [4.62] and [4.63]):

$$u_1' = \cos\theta - 1, \qquad u_2' = \sin\theta \tag{4.78}$$

Hence, we can assume θ as the master variable, and consider u_1, u_2 as slave variables. This choice has the advantage that the unique admissible strain is linear in the master displacement, namely $\kappa = \theta'$. Since the slave variables are expressed in differential form, they cannot be condensed, but the constraints must be accounted as auxiliary conditions. By time-differentiating the latter, it follows:

$$\dot{u}_1' = -\dot{\theta}\sin\theta, \qquad \dot{u}_2' = \dot{\theta}\cos\theta \tag{4.79}$$

Since $\dot{\kappa} = \dot{\theta}'$, $\omega = \dot{\theta}$, the VPP becomes:

$$\begin{aligned} \int_{\mathcal{S}} M\dot{\theta}' ds &= \int_{\mathcal{S}} \left(\bar{p}_1\dot{u}_1 + \bar{p}_2\dot{u}_2 + \bar{c}\dot{\theta}\right) ds + \sum_{H=A}^{B} \left[\bar{P}_1\dot{u}_1 + \bar{P}_2\dot{u}_2 + \bar{C}\dot{\theta}\right]_H \\ &- \int_{\mathcal{S}} \left[\lambda_1\left(\dot{u}_1' + \dot{\theta}\sin\theta\right) + \lambda_2\left(\dot{u}_2' - \dot{\theta}\cos\theta\right)\right] ds \qquad \forall\left(\dot{u}_1, \dot{u}_2, \dot{\theta}\right) \end{aligned} \tag{4.80}$$

where λ_1, λ_2 are Lagrangian multipliers. By performing just one integration by parts, the VPP yields the following field equations:

$$\begin{aligned} &\lambda_1' + \bar{p}_1 = 0 \\ &\lambda_2' + \bar{p}_2 = 0 \\ &M' + (-\lambda_1\sin\theta + \lambda_2\cos\theta) + \bar{c} = 0 \end{aligned} \tag{4.81}$$

and the alternative boundary conditions:

$$\begin{aligned} &\left[\bar{P}_1 \pm \lambda_1\right]_H \dot{u}_{1H} = 0 \\ &\left[\bar{P}_2 \pm \lambda_2\right]_H \dot{u}_{2H} = 0 \\ &\left[\bar{C} \pm M\right]_H \dot{\theta}_H = 0 \end{aligned} \tag{4.82}$$

When the bending moment is expressed as $M = EJ\kappa = EJ\theta'$ and the d'Alembert principle is used, the whole elastic problem becomes:

$$\begin{aligned} &EJ\theta'' + (-\lambda_1\sin\theta + \lambda_2\cos\theta) + \bar{c} = 0 \\ &\lambda_1' + \bar{p}_1 = m\ddot{u}_1 \\ &\lambda_2' + \bar{p}_2 = m\ddot{u}_2 \\ &u_1' = \cos\theta - 1 \\ &u_2' = \sin\theta \end{aligned} \tag{4.83}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with the alternative mechanical boundary conditions:

$$\mp EJ\theta'_H = \bar{C}_H$$
$$\mp \lambda_{1H} = \bar{P}_{1H} \quad [4.84]$$
$$\mp \lambda_{2H} = \bar{P}_{2H}$$

and/or the geometric boundary conditions:

$$u_{1H} = \breve{u}_{1H}, \quad u_{2H} = \breve{u}_{2H}, \quad \theta_H = \breve{\theta}_H \quad [4.85]$$

4.6.4 The direct condensation of the elastica equilibrium equations

In the previous sections, we derived the condensed balance (or equilibrium) equations by the VPP, in the frameworks of the hybrid or mixed formulations. However, balance equations for the unconstrained beam are known, and the filtering of the reactive stresses can also be performed in a direct way, by suitable algebraic-differential combinations of the field equations. Now, we want to follow the latter approach, aimed to throw light on the different mechanical meaning of the Lagrange multipliers we encountered in our treatment.

To this end, we reconsider the equilibrium equations [2.103]:

$$\mathbf{t}' + \mathbf{p} = \mathbf{0}$$
$$\mathbf{m}' + \mathbf{a}_1 \times \mathbf{t} + \mathbf{c} = \mathbf{0} \quad [4.86]$$

in which, by virtue of unshearability and inextensibility of the elastica, we substituted $\mathbf{x}' = \mathbf{a}_1$. In these equations, $\mathbf{m} = M\mathbf{a}_3$ is the unique active stress, while $\mathbf{t}$ is of reactive type, since $\dot{\mathbf{e}} = \mathbf{0}$ in each admissible motion. However, we have several possibilities to represent $\mathbf{t}$ by components, of course all equivalent. Among the infinite, we consider the following three representations (see Figure 4.2):

$$\mathbf{t} = N\mathbf{a}_1 + T\mathbf{a}_2$$
$$\mathbf{t} = R_1\mathbf{a}_1 + R_2\bar{\mathbf{a}}_2 \quad [4.87]$$
$$\mathbf{t} = S_1\bar{\mathbf{a}}_1 + S_2\bar{\mathbf{a}}_2$$

The first and third choices appear the more "natural", since the components are evaluated in an orthogonal basis, intrinsic to the beam (N, T), or extrinsic (S_1, S_2); the second choice, instead, is less obvious, since the components (R_1, R_2) are expressed in a *non-orthogonal* basis; nonetheless, we will prove that all three representations are meaningful.

Now, we will project the equilibrium equations always in the reference basis $\bar{\mathcal{B}}$, but adopt different representations for the reactive stress.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

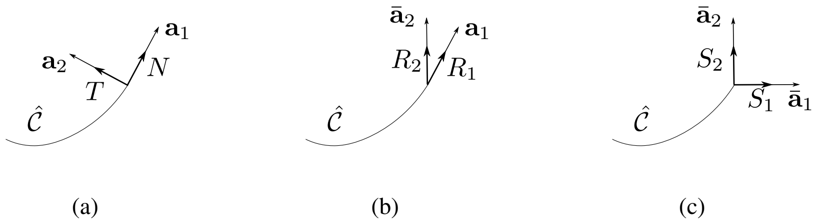


Figure 4.2: Alternative representations of the reactive force-stress of the elastica: (a) in the intrinsic basis, (b) in a non-orthogonal basis, (c) in the extrinsic basis.

Intrinsic components

By using the components of $\mathbf{t}$ in the $\mathcal{B}$-basis, the equilibrium equations in $\bar{\mathcal{B}}$ become:

$$\begin{aligned} &(N\cos\theta - T\sin\theta)' + \bar{p}_1 = 0 \\ &(N\sin\theta + T\cos\theta)' + \bar{p}_2 = 0 \\ &M' + T + \bar{c} = 0 \end{aligned} \qquad [4.88]$$

The shear force can be condensed by evaluating it from the third equation, and substituting it in the first two equations, thus obtaining:

$$\begin{aligned} &[N\cos\theta + (M' + \bar{c})\sin\theta]' + \bar{p}_1 = 0 \\ &[N\sin\theta - (M' + \bar{c})\cos\theta]' + \bar{p}_2 = 0 \end{aligned} \qquad [4.89]$$

These equations coincide with equations [4.48], then reobtained in the Version I of the hybrid formulation. Therefore, the Lagrange multiplier used there is indeed the normal force N. The other Lagrange multiplier T, in contrast, did not appear there, since we used unshearability to condense the rotation, instead to append it.

Oblique components

By using the components of $\mathbf{t}$ in the non-orthogonal $(\mathbf{a}_1, \bar{\mathbf{a}}_2)$-basis, the equilibrium equations in $\bar{\mathcal{B}}$ become:

$$\begin{aligned} &(R_1\cos\theta)' + \bar{p}_1 = 0 \\ &(R_1\sin\theta + R_2)' + \bar{p}_2 = 0 \\ &M' + R_2\cos\theta + \bar{c} = 0 \end{aligned} \qquad [4.90]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The R_2-component can be condensed after having been evaluated by the third equation and substituted in the remaining equations:

$$(R_1 \cos\theta)' + \bar{p}_1 = 0$$
$$\left[R_1 \sin\theta - \frac{M' + \bar{c}}{\cos\theta}\right]' + \bar{p}_2 = 0 \tag{4.91}$$

These equations coincide with that obtained in the hybrid formulation (equations [4.73], in the Version II, when $\lambda \equiv R_1$. Therefore, we conclude that *the Lagrange multiplier* λ *assumes the meaning of the along-axis internal force*, when the other component is directed along $\bar{\mathbf{a}}_2$ (i.e. R_1 is *not* the normal force). In other words, R_1, R_2 are the reactive forces dual of the geometrical constraints enforced.

Extrinsic components

By using the components of $\mathbf{t}$ in the $\bar{\mathcal{B}}$-basis, the equilibrium equations in the same basis become:

$$S_1' + \bar{p}_1 = 0$$
$$S_2' + \bar{p}_2 = 0 \tag{4.92}$$
$$M' + S_2 \cos\theta - S_1 \sin\theta + \bar{c} = 0$$

These coincide with equations [4.81], obtained by the mixed formulation, with $\lambda_1 \equiv S_1$, $\lambda_2 \equiv S_2$. Therefore, the Lagrangian multipliers assume the meaning of extrinsic components of the reactive force.

4.7 The inextensible, unshearable and untwistable straight beam

We consider a highly internally constrained model of beam, for which twist, in addition to shear and extension, is prevented. Since such a beam can only undergo flexure in two planes, it could be named *flexural–flexural beam* for short. As we noted in the general discussion, this is the case of compact or boxed cross-sections. Of course, a beam can be untwistable but extensible; however, we will skip this case by leaving the problem to the reader.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Kinematics

There are four constraints to be satisfied, namely $\varepsilon = \gamma_2 = \gamma_3 = 0$ and $\kappa_1 = 0$. The first three conditions lead to equations already obtained for Euler's elastica (Version II, equation [4.57], and equation [4.54]) and the last condition follows from the first row of equation [2.53]; overall:

$$\begin{aligned}
&\tan\theta_2 = -\frac{u_3'}{\sqrt{1-u_3'^2}} \\
&\tan\theta_3 = \frac{u_2'}{\sqrt{1-u_2'^2-u_3'^2}} \\
&\varepsilon = \left(1+u_1'\right)^2 + u_2'^2 + u_3'^2 - 1 = 0 \\
&\kappa_1 = \theta_1' - \theta_3' \sin\theta_2 = 0
\end{aligned} \tag{4.93}$$

According to the hybrid formulation, we will condense the rotations θ_2, θ_3, while we will account for inextensibility and untwistability via Lagrange multipliers. The admissible strains reduce to the two flexural curvatures κ_2, κ_3, given by the second and third rows of equation [2.53].

Virtual Power Principle

The VPP becomes:

$$\begin{aligned}
\int_0^{\mathcal{S}} \left(M_2\dot{\kappa}_2 + M_3\dot{\kappa}_3\right) ds = &\int_{\mathcal{S}} \left(\dot{\boldsymbol{u}}^T\bar{\boldsymbol{p}} + \dot{\boldsymbol{\theta}}^T\bar{\boldsymbol{B}}_\omega^T\bar{\boldsymbol{c}}\right) ds \\
&+ \sum_{H=A}^{B}\left[\dot{\boldsymbol{u}}^T\bar{\boldsymbol{P}} + \dot{\boldsymbol{\theta}}^T\bar{\boldsymbol{B}}_\omega^T\bar{\boldsymbol{C}}\right]_H - \int_{\mathcal{S}} \left[\lambda\dot{\varepsilon} + \mu_1\dot{\kappa}_1\right] ds
\end{aligned} \tag{4.94}$$

where λ, μ_1 are two Lagrange multipliers. This equation coincides with equation [4.31], valid for an extensible and twistable beam, with λ replacing N and μ_1 replacing M_1. Therefore, four balance equations are obtained, formally identical to equations [4.35] and [4.36].

The Fundamental Problem

By expressing the active stresses as $M_2 = \overset{\circ}{M}_2 + EJ_2\kappa_2$, $M_3 = \overset{\circ}{M}_3 + EJ_3\kappa_3$ and κ_2, κ_3 in terms of the master displacements, we obtain a system of four balance equations and two geometrical constraints in the four displacement unknowns $\boldsymbol{u}, \theta_1$ and the two reactive stresses λ, μ_1.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

4.8 The foil-beam

We want to formulate a model for an unshearable, inextensible and one-plane-inflexible beam, embedded in a 3D-space; in short, a *lamina*, or *foil*-beam.

Kinematics

The beam must obey four constraint conditions, $\varepsilon = \gamma_2 = \gamma_3 = 0$ and $\kappa_3 = 0$ (if $J_3 \gg J_2$). The first three conditions have already been accounted for in the biflexible model and the last condition follows from the third row of equation [2.53]; overall:

$$\begin{aligned}
\tan\theta_2 &= -\frac{u_3'}{\sqrt{1-u_3'^2}} \\
\tan\theta_3 &= \frac{u_2'}{\sqrt{1-u_2'^2-u_3'^2}} \\
\varepsilon &= \left(1+u_1'\right)^2 + u_2'^2 + u_3'^2 - 1 = 0 \\
\kappa_1 &= \theta_1' - \theta_3' \sin\theta_2 = 0 \\
\kappa_3 &= -\theta_2' \sin\theta_1 + \theta_3' \cos\theta_1 \cos\theta_2 = 0
\end{aligned} \qquad [4.95]$$

According to the hybrid formulation, we will condense the rotations θ_2, θ_3, while we will account for inextensibility, untwistability and one-plane-inflexibility via Lagrange multipliers. The admissible strain reduces to just a flexural curvature κ_2, given by equation [2.53b].

Virtual Power Principle

The VPP becomes:

$$\begin{aligned}
\int_{\mathcal{S}} M_2 \dot{\kappa}_2 ds &= \int_{\mathcal{S}} \left(\dot{\boldsymbol{u}}^T \bar{\boldsymbol{p}} + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{c}}\right) ds \\
&+ \sum_{H=A}^{B} \left[\dot{\boldsymbol{u}}^T \bar{\boldsymbol{P}} + \dot{\boldsymbol{\theta}}^T \bar{\boldsymbol{B}}_\omega^T \bar{\boldsymbol{C}}\right]_H - \int_{\mathcal{S}} \left[\lambda\dot{\varepsilon} + \mu_1\dot{\kappa}_1 + \mu_3\dot{\kappa}_3\right] ds
\end{aligned} \qquad [4.96]$$

where λ, μ_1, μ_3 are Lagrange multipliers. This equation coincides with equation [4.31], valid for an extensible, twistable and flexible beam, with λ, μ_1, μ_3 replacing N, M_1, M_3, in the order. Therefore, four balance equations are obtained, identical to equations [4.35] and [4.36].

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The Fundamental Problem

By expressing the active stress as $M_2 = \overset{\circ}{M}_2 + EJ_2\kappa_2$, and κ_3 in terms of the master displacements, we obtain a system of four balance equations and three geometrical constraints in the displacement unknowns $\boldsymbol{u}$, θ_1 and in the reactive stresses λ, μ_1, μ_3.

4.9 The shear–shear–torsional beam

A shear-beam is a beam that experiences shear strains much larger than flexural-strains. This model is often adopted in seismology to study the propagation of transverse- (or shear-) waves through stratified (and often inhomogeneous) layers. In structural engineering, this is a homogeneous (and coarse) model for planar shear-type frames, under planar excitation transverse to the axis (e.g. for seismic or wind excitation). Here, the macroscopic shear strain is indeed produced by bending of the columns, accompanied by negligible rotations of the (assumed) rigid floors, since rotations are prevented by the high axial stiffness of the columns.

A 3D-model of shear-beam could also be used to (roughly) analyze a *tower-building*, by including the torsional effect induced by the rotations of the floors around the tower axis. In this case, we have a *shear–shear–torsional* beam, whose model we want to derive now. We will follow the mixed formulation, by specifically referring to a *cantilevered beam* (i.e. the model of tower-building), for which analysis strongly simplifies.

Kinematics

We start from the unconstrained model by enforcing three internal constraints, prescribing the vanishing of the bending curvatures, $\kappa_2 = \kappa_3 = 0$, and the vanishing of the extension, $\varepsilon = 0$. By remembering the strain–displacement relationships [2.53b,c], we see that inflexibility entails that $\theta_2' = \theta_3' = 0$, i.e. $\theta_2 =$ const, $\theta_3 =$ const. By accounting for the clamp boundary condition at the end A, we have:

$$\theta_2 \equiv 0, \quad \theta_3 \equiv 0 \qquad [4.97]$$

Consequently, from equation [2.52a], the unit extension $\varepsilon = u_1'$ follows; inextensibility therefore entails $u_1 =$ const, or by using the clamp condition:

$$u_1 \equiv 0 \qquad [4.98]$$

In conclusion, the cantilever shear–shear–torsional beam only experiences transverse displacements u_2, u_3 and a twist $\theta := \theta_1$; moreover, from equations [2.71a] and [2.73]), the unique non-zero spin component is $\bar{\omega}_1 = \dot{\theta}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

With the previous results, the admissible strains (equation [2.53a] and [2.52b,c]), assume the following simple forms:

$$\begin{aligned}\kappa_1 &= \theta' \\ \gamma_2 &= u_2' \cos\theta + u_3' \sin\theta \\ \gamma_3 &= -u_2' \sin\theta + u_3' \cos\theta\end{aligned} \qquad [4.99]$$

Geometrical boundary conditions at the clamped end A require:

$$u_{2A} = u_{3A} = \theta_A = 0 \qquad [4.100]$$

Time-differentiation of the admissible strains provides:

$$\begin{aligned}\dot{\kappa}_1 &= \dot{\theta}' \\ \dot{\gamma}_2 &= \dot{u}_2' \cos\theta + \dot{u}_3' \sin\theta + \dot{\theta}(-u_2' \sin\theta + u_3' \cos\theta) \\ \dot{\gamma}_3 &= -\dot{u}_2' \sin\theta + \dot{u}_3' \cos\theta - \dot{\theta}(u_2' \cos\theta + u_3' \sin\theta)\end{aligned} \qquad [4.101]$$

Virtual Power Principle

The VPP [4.6], by accounting for $\dot{\varepsilon} = \dot{\kappa}_2 = \dot{\kappa}_3 = 0$, becomes:

$$\begin{aligned}&\int_{\mathcal{S}} (T_2\dot{\gamma}_2 + T_3\dot{\gamma}_3 + M_1\dot{\kappa}_1)\, ds \\ &\quad = \int_{\mathcal{S}} \left(\bar{p}_2\dot{u}_2 + \bar{p}_3\dot{u}_3 + \bar{c}\dot{\theta}\right) ds + \sum_{H=A}^{B} \left[\bar{P}_2\dot{u}_2 + \bar{P}_3\dot{u}_3 + \bar{C}\dot{\theta}\right]_H\end{aligned} \qquad [4.102]$$

where the index 1 has been omitted on the couples. After substitution of the strain-rates [4.101] and integration by parts, the balance equations follow:

$$\begin{aligned}&(T_2 \cos\theta - T_3 \sin\theta)' + \bar{p}_2 = 0 \\ &(T_2 \sin\theta + T_3 \cos\theta)' + \bar{p}_3 = 0 \\ &M_1' + T_2(u_2' \sin\theta - u_3' \cos\theta) + T_3(u_2' \cos\theta + u_3' \sin\theta) + \bar{c}_1 = 0\end{aligned} \qquad [4.103]$$

with the boundary conditions, holding at the free end B:

$$\begin{aligned}&\left[T_2 \cos\theta - T_3 \sin\theta - \bar{P}_2\right]_B = 0 \\ &\left[T_2 \sin\theta + T_3 \cos\theta - \bar{P}_3\right]_B = 0 \\ &\left[M_1 - \bar{C}_1\right]_B = 0\end{aligned} \qquad [4.104]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Elastic law

We assume a linear elastic law for active stresses and admissible strains. However, the diagonal form we extensively used so far, referred to compact cross-sections beams, is not suited to the scope when we deal with a tower-building model. For non-compact, non-symmetric cross-sections, indeed, the *flexural-center*[11] could be far from the centroid, making the uncoupling inaccurate. Therefore, the full-matrix Hooke law equation [2.157], must be referred to.

We define the flexural-center of the cross-section by point $C := (x_{2C}, x_{3C})$, having the property that, in linear elasticity, any shear force $\mathbf{t}_C := T_2\bar{\mathbf{a}}_2 + T_3\bar{\mathbf{a}}_3$ applied to it produces shear strains γ_2, γ_3, *without torsion,* $\kappa_1 = 0$. Consequently, from Maxwell's theorem, the flexural-center is also the *torsional-center*, since a twist-moment $\mathbf{m} = M_1\bar{\mathbf{a}}_1$ induces a rotation around C[12]. Because of these properties, we adopt the following elastic potential:

$$\begin{aligned}\phi :=& \mathring{M}_1\kappa_1 + \mathring{T}_2\gamma_2 + \mathring{T}_3\gamma_3 \\ &+ \frac{1}{2}\left[GJ_C\kappa_1^2 + GA_2\left(\gamma_2 - \kappa_1 x_{3C}\right)^2 + GA_3\left(\gamma_3 + \kappa_1 x_{2C}\right)^2\right]\end{aligned} \qquad [4.105]$$

where GJ_C is the *torsional stiffness* (evaluated with respect to C), and GA_2, GA_3 are the *shear stiffnesses*; moreover, over-ringed quantities denote an assumed known prestress existing in the reference configuration. It should be noted that the potential is such that the shear contribution vanishes when $\gamma_2 - \kappa_1 x_{3C} = 0$, $\gamma_3 + \kappa_1 x_{2C} = 0$, i.e. when (in the linear approximation) C is the rotation center. This potential, when the Green law is used:

$$M_1 = \frac{\partial\phi}{\partial\kappa_1}, \quad T_2 = \frac{\partial\phi}{\partial\gamma_2}, \quad T_3 = \frac{\partial\phi}{\partial\gamma_3} \qquad [4.106]$$

leads to:

$$\begin{pmatrix} M_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} \mathring{M}_1 \\ \mathring{T}_2 \\ \mathring{T}_3 \end{pmatrix} + \begin{bmatrix} GJ_G & -GA_2x_{3C} & GA_3x_{2C} \\ -GA_2x_{3C} & GA_2 & 0 \\ GA_3x_{2C} & 0 & GA_3 \end{bmatrix} \begin{pmatrix} \kappa_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \qquad [4.107]$$

where[13]:

$$GJ_G := GJ_C + GA_2x_{3C}^2 + GA_3x_{2C}^2 \qquad [4.108]$$

is the *torsional stiffness with respect to the centroid.*

---

11. Also known as the *shear-center*, or, when referred to a structure as the tower-building, the *stiffness center*.

12. The translation of C produced by the twist-moment equates the rotation produced by $\mathbf{t}_C$, i.e. zero.

13. We know that when a pure shear strain $\gamma_2\bar{\mathbf{a}}_2 + \gamma_3\bar{\mathbf{a}}_3$ is assigned to the beam, a shear force $\mathbf{t}_C = GA_2\gamma_2\bar{\mathbf{a}}_2 + GA_3\gamma_3\bar{\mathbf{a}}_3$, applied to the flexural center, and a couple $\mathbf{m} = \mathbf{0}$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Elastic constants identification

The three elastic constants GJ_C, GA_2, GA_3 could be identified by the (refined) model of 3D-frame via a linearized kinematic analysis and an equivalence condition in energy. To this end, a *cell*, made up of two adjacent rigid floors, connected by elastic columns of equal height h is considered. An (infinitely small) relative motion between the two floors is considered of components $\Delta u_2 = \gamma_2 h$, $\Delta u_3 = \gamma_3 h$, $\Delta\theta = \kappa_1 h$. The end-sections of a column located at x_{2i}, x_{3i} undergo a relative displacement:

$$\Delta u_{2i} = h\left(\gamma_2 - \kappa_1 x_{3i}\right), \quad \Delta u_{3i} = h\left(\gamma_3 + \kappa_1 x_{2i}\right), \quad \Delta\theta_i = h\kappa_1 \qquad [4.109]$$

so that the ith column stores the elastic energy:

$$U_i := \frac{1}{2}\left(K_{1i}\Delta\theta_i^2 + K_{2i}\Delta u_{2i}^2 + K_{3i}\Delta u_{3i}^2\right) =: U_i\left(\kappa_1, \gamma_2, \gamma_3\right) \qquad [4.110]$$

where $K_{1i} := GJ_{1i}/h$, $K_{2i} := 12EJ_{3i}/h^3$, $K_{3i} := 12EJ_{2i}/h^3$ are its torsional and flexural stiffnesses. By summing up the energies stored by N columns, $U := \sum_{i=1}^{N} U_i$ is obtained. After that, a *potential per unit length* can be defined for the frame as $\phi = U\left(\kappa_1, \gamma_2, \gamma_3\right)/h$. This expression is finally adopted for the rough model, from which the stresses are derived via the Green law.

The Fundamental Problem

The Fundamental Problem for the shear–shear beam is governed by three balance equations [4.103], the three unconstrained strain–displacement relationships [4.99], and the three elastic laws [4.107], in the active stresses T_2, T_3, M_1, three unconstrained strains $\gamma_2, \gamma_3, \kappa_1$ and three displacements u_2, u_3, θ_1, all unknown.

The planar model: the shear-beam

When the beam is planar, we have $u_3 = 0$, $\theta = 0$. The only strain different from zero is $\gamma_2 = u_2'$, which, therefore, is linear in the displacement. The relevant balance

---

arise; moreover, when the beam undergoes a pure torsion $\kappa_1 \bar{\mathbf{a}}_1$ around the flexural axis, a couple $\mathbf{m} = GJ_C\kappa_1\bar{\mathbf{a}}_1$ and a shear force $\mathbf{t}_C = \mathbf{0}$ arise. Equation [4.107] express just these properties; however, T_2, T_3 are the components of the shear force $\mathbf{t}_G$ applied to the centroid (not at C!), and M_1 is the torsional moment evaluated with respect to the same point and the strains are the dual kinematic quantities. As a matter of fact, if the strains $(0, \gamma_2, \gamma_3)$ are enforced, according to equation [4.107], the force $\mathbf{t}_G := GA_2\gamma_2\bar{\mathbf{a}}_2 + GA_3\gamma_3\bar{\mathbf{a}}_3$ and the couple $\mathbf{m} = \left(-GA_2\gamma_2 x_{3C} + GA_3\gamma_3 x_{3C}\right)\bar{\mathbf{a}}_1$ arise, equivalent to $\mathbf{t}_C = \mathbf{t}_G$ applied to the flexural center. If the strains $\left(\kappa_1, \kappa_1 x_{3C}, -\kappa_1 x_{2C}\right)$ are assigned, equivalent to a torsion around C, then the couple $\mathbf{m} = \left(GJ_G - GA_2 x_{3C}^2 - GA_3 x_{2C}^2\right)\kappa_1\bar{\mathbf{a}}_1 = GJ_C\kappa_1\bar{\mathbf{a}}_1$ and the force $\mathbf{t}_G = \mathbf{0}$ are obtained.

equation, equation [4.103b], also linearizes, becoming $T_2' + p_2 = m\ddot{u}_2$. By taking $T_2 = \mathring{T}_2 + GA_2\gamma_2$ and $p_2 = \mathring{p}_2 + \tilde{p}_2$, with $\mathring{T}_2' + \mathring{p}_2$=0, the balance equation becomes:

$$GA_2 u_2'' + \tilde{p}_2 = 0 \qquad [4.111]$$

i.e. it is linear.

4.10 The planar unshearable and inextensible curved beam

When the beam is curved, equations are, of course, much more cumbersome, but the methods we have illustrated for the straight beam still hold. As an example, we derive here the model of a planar curved beam, under the hypotheses of the elastica, i.e. unsherability and inextensibility. We will illustrate both the hybrid and mixed formulations, thus generalizing the results we achieved for the straight beam.

4.10.1 The hybrid formulation

Kinematics

We know, from equation [3.72], that strains for a curved planar beam are:

$$\begin{aligned} \varepsilon &= -1 + (1 + u_1' - \bar{\kappa} u_2)\cos\theta + (u_2' + \bar{\kappa} u_1)\sin\theta \\ \gamma &= -(1 + u_1' - \bar{\kappa} u_2)\sin\theta + (u_2' + \bar{\kappa} u_1)\cos\theta \\ \chi &= \theta' \end{aligned} \qquad [4.112]$$

By enforcing $\gamma = 0$, squaring and summing equation [4.112a,b], we obtain ε; by vanishing it, we have:

$$\varepsilon = \sqrt{(1 + u_1' - \bar{\kappa} u_2)^2 + (u_2' + \bar{\kappa} u_1)^2} - 1 = 0 \qquad [4.113]$$

Then, the rotation angle follows from equation [4.112b]:

$$\theta = \arctan \frac{u_2' + \bar{\kappa} u_1}{\sqrt{1 - (u_2' + \bar{\kappa} u_1)^2}} \qquad [4.114]$$

in which equation [4.113] has been used. Equivalently:

$$\cos\theta = \sqrt{1 - (u_2' + \bar{\kappa} u_1)^2}, \qquad \sin\theta = u_2' + \bar{\kappa} u_1 \qquad [4.115]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Equations [4.113] and [4.114] generalize equation [4.68], valid for the straight planar beam, and used in the framework of Version II of the hybrid formulation. By time-differentiating them, we obtain constraints for velocities:

$$\begin{aligned} \dot{\theta} &= \frac{\bar{\kappa}}{\sqrt{1-(u_2'+\bar{\kappa}u_1)^2}}\dot{u}_1 + \frac{1}{\sqrt{1-(u_2'+\bar{\kappa}u_1)^2}}\dot{u}_2' \\ \dot{\varepsilon} &= \bar{\kappa}\dot{u}_1\sin\theta - \bar{\kappa}\dot{u}_2\cos\theta + \dot{u}_1'\cos\theta + \dot{u}_2'\sin\theta = 0 \end{aligned} \quad [4.116]$$

in which sine and cosine of θ must be considered expressed by equation [4.115]. Finally, $\dot{\chi} = \dot{\theta}'$ and $\bar{\omega} = \dot{\theta}$. According to the hybrid formulation, we will condense the rotation, and account for inextensibility via a Lagrange multiplier.

Virtual Power Principle

The VPP, fulfilling to inextensibility, becomes (as in equation [4.6]) (with $\dot{\chi}$ replacing $\dot{\kappa}$):

$$\begin{aligned} \int_{\mathcal{S}} M\dot{\chi}ds &= \int_{\mathcal{S}} \left(\bar{p}_1\dot{u}_1 + \bar{p}_2\dot{u}_2 + \bar{c}\dot{\theta}\right) ds \\ &+ \sum_{H=A}^{B}\left[\bar{P}_1\dot{u}_1 + \bar{P}_2\dot{u}_2 + \bar{C}\dot{\theta}\right]_H - \int_{\mathcal{S}} \lambda\dot{\varepsilon}ds \end{aligned} \quad [4.117]$$

After substitution for $\dot{\chi}, \dot{\varepsilon}$ and a first integration by parts, we have:

$$\begin{aligned} &-\int_{\mathcal{S}} M'\dot{\theta}ds + \left[M\dot{\theta}\right]_A^B = \int_{\mathcal{S}} \left(\bar{p}_1\dot{u}_1 + \bar{p}_2\dot{u}_2 + \bar{c}\dot{\theta}\right) ds \\ &-\int_{\mathcal{S}} \left[\left(\lambda\bar{\kappa}\sin\theta - (\lambda\cos\theta)'\right)\dot{u}_1 - \left(\lambda\bar{\kappa}\cos\theta - (\lambda\sin\theta)'\right)\dot{u}_2\right] ds \\ &-\left[\lambda\cos\theta\dot{u}_1 + \lambda\sin\theta\dot{u}_2\right] + \sum_{H=A}^{B}\left[\bar{P}_1\dot{u}_1 + \bar{P}_2\dot{u}_2 + \bar{C}\dot{\theta}\right]_H \end{aligned} \quad [4.118]$$

After substitution for $\dot{\theta}$ under the signs of integral, and a second integration by parts, the balance equations are finally obtained:

$$\begin{aligned} &-\lambda\bar{\kappa}\sin\theta + (\lambda\cos\theta)' + \frac{\bar{\kappa}(M'+c_1)}{\sqrt{1-(u_2'+\bar{\kappa}u_1)^2}} - \bar{p}_1 = 0 \\ &\lambda\bar{\kappa}\cos\theta - (\lambda\sin\theta)' - \left[\frac{(M'+c_1)}{\sqrt{1-(u_2'+\bar{\kappa}u_1)^2}}\right]' - \bar{p}_2 = 0 \end{aligned} \quad [4.119]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with the mechanical boundary conditions:

$$\begin{aligned}
&\left[\bar{P}_1 \pm \lambda\cos\theta\right]_H \dot{u}_{1H} = 0\\
&\left[\bar{P}_2 \pm \left(\lambda\sin\theta + \frac{(M' + c_1)}{\sqrt{1-(u_2' + \bar{\kappa}u_1)^2}}\right)\right]_H \dot{u}_{2H} = 0\\
&\left[\bar{C} \pm M\right]_H \dot{\theta}_H = 0
\end{aligned} \qquad [4.120]$$

These equations generalize equations [4.73] and [4.74], obtained for the straight beam.

The Fundamental Problem

Once the bending moment is expressed via the elastic law, and then by displacements, and the d'Alembert principle is used, the Fundamental Problem turns out to be governed by (compare with equations [4.75] and [4.76]):

$$\begin{aligned}
&-\lambda\bar{\kappa}(u_2' + \bar{\kappa}u_1) + \left(\lambda\sqrt{1-(u_2' + \bar{\kappa}u_1)^2}\right)'\\
&\quad + \frac{\bar{\kappa}}{\sqrt{1-(u_2' + \bar{\kappa}u_1)^2}}\left[\left(EJ\left(\frac{u_2'' + \bar{\kappa}u_1'}{\sqrt{1-(u_2' + \bar{\kappa}u_1)^2}}\right)\right)' + \bar{c}\right]\\
&\quad - \bar{p}_1 + m\ddot{u}_1 = 0\\
&\lambda\bar{\kappa}\sqrt{1-(u_2' + \bar{\kappa}u_1)^2} - (\lambda(u_2' + \bar{\kappa}u_1))'\\
&\quad - \left[\frac{1}{\sqrt{1-(u_2' + \bar{\kappa}u_1)^2}}\left[\left(EJ\left(\frac{u_2'' + \bar{\kappa}u_1'}{\sqrt{1-(u_2' + \bar{\kappa}u_1)^2}}\right)\right)' + \bar{c}\right]\right]'\\
&\quad - \bar{p}_2 + m\ddot{u}_2 = 0
\end{aligned} \qquad [4.121]$$

and, on the boundary:

$$\begin{aligned}
&\mp\left[\lambda\sqrt{1-(u_2' + \bar{\kappa}u_1)^2}\right]_H = \bar{P}_{1H}\\
&\mp\left[\lambda(u_2' + \bar{\kappa}u_1)\right.\\
&\quad \left. + \frac{1}{\sqrt{1-(u_2' + \bar{\kappa}u_1)^2}}\left[\left(EJ\frac{u_2'' + \bar{\kappa}u_1'}{\sqrt{1-(u_2' + \bar{\kappa}u_1)^2}}\right)' + \bar{c}\right]\right]_H = \bar{P}_{2H}\\
&\mp\left[EJ\frac{u_2'' + \bar{\kappa}u_1'}{\sqrt{1-(u_2' + \bar{\kappa}u_1)^2}}\right]_H = \bar{C}_H
\end{aligned} \qquad [4.122]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

together with:

$$u_{1H} = \breve{u}_{1H}, \quad u_{2H} = \breve{u}_{2H}, \quad \left[\arctan \frac{u_2'}{\sqrt{1-u_2'^2}}\right]_H = \breve{\theta}_{3H} \qquad [4.123]$$

Moreover, the inextensibility condition [4.113] must be appended.

4.10.2 The mixed formulation

As for the straight beam, we can formulate the problem by renouncing to condense the rotation, but rather enforcing both unshearability and inextensibility constrains via Lagrange multipliers.

First, we rewrite the two conditions in the form:

$$u_1' = \cos\theta - 1 + \bar{\kappa} u_2, \qquad u_2' = \sin\theta - \bar{\kappa} u_1 \qquad [4.124]$$

which generalize equations [4.78]. Then, we time-differentiate them, to obtain:

$$\dot{u}_1' = -\dot{\theta}\sin\theta + \bar{\kappa}\dot{u}_2, \qquad \dot{u}_2' = \dot{\theta}\cos\theta - \bar{\kappa}\dot{u}_1 \qquad [4.125]$$

Since $\dot{\chi} = \dot{\theta}'$, $\bar{\omega} = \dot{\theta}$, the VPP becomes:

$$\int_{\mathcal{S}} M\dot{\theta}' ds = \int_{\mathcal{S}} \left(\bar{p}_1\dot{u}_1 + \bar{p}_2\dot{u}_2 + \bar{c}\dot{\theta}\right) ds + \sum_{H=A}^{B} \left[\bar{P}_1\dot{u}_1 + \bar{P}_2\dot{u}_2 + \bar{C}\dot{\theta}\right]_H$$
$$- \int_{\mathcal{S}} \left[\lambda_1\left(\dot{u}_1' + \dot{\theta}\sin\theta - \bar{\kappa}\dot{u}_2\right) + \lambda_2\left(\dot{u}_2' - \dot{\theta}\cos\theta + \bar{\kappa}\dot{u}_1\right)\right] ds \qquad [4.126]$$
$$\forall \left(\dot{u}_1, \dot{u}_2, \dot{\theta}\right)$$

where λ_1, λ_2 are Lagrangian multipliers. By performing just one integration by parts, the VPP yields the following field equations:

$$\begin{aligned} &\lambda_1' - \bar{\kappa}\lambda_2 + \bar{p}_1 = 0 \\ &\lambda_2' + \bar{\kappa}\lambda_1 + \bar{p}_2 = 0 \\ &M' + (-\lambda_1\sin\theta + \lambda_2\cos\theta) + \bar{c} = 0 \end{aligned} \qquad [4.127]$$

and the alternative boundary conditions:

$$\begin{aligned} &\left[\bar{P}_1 \pm \lambda_1\right]_H \dot{u}_{1H} = 0 \\ &\left[\bar{P}_2 \pm \lambda_2\right]_H \dot{u}_{2H} = 0 \\ &\left[\bar{C} \pm M\right]_H \dot{\theta}_H = 0 \end{aligned} \qquad [4.128]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

When the bending moment is expressed as $M = EJ\kappa = EJ\theta'$ and the d'Alembert principle is used, the whole elastic problem becomes (compare with equations [4.83] and [4.84]):

$$\begin{aligned} &EJ\theta'' + (-\lambda_1 \sin\theta + \lambda_2 \cos\theta) + \bar{c} = 0 \\ &\lambda_1' - \bar{\kappa}\lambda_2 + \bar{p}_1 = m\ddot{u}_1 \\ &\lambda_2' + \bar{\kappa}\lambda_1 + \bar{p}_2 = m\ddot{u}_2 \\ &u_1' = \cos\theta - 1 + \bar{\kappa}u_1 \\ &u_2' = \sin\theta - \bar{\kappa}u_2 \end{aligned} \qquad [4.129]$$

with the alternative mechanical boundary conditions:

$$\begin{aligned} &\mp \lambda_1 = \bar{P}_{1H} \\ &\mp \lambda_2 = \bar{P}_{2H} \\ &\mp EJ\theta' = \bar{C}_H \end{aligned} \qquad [4.130]$$

and/or the geometric boundary conditions:

$$u_{1H} = \breve{u}_{1H}, \quad u_{2H} = \breve{u}_{2H}, \quad \theta_H = \breve{\theta}_H \qquad [4.131]$$

4.11 Summary

In this chapter, we developed specific models for constrained beams. Initially, we discussed the conditions under which a beam can be modeled as internally constrained. We wrote the elastic potential for an unconstrained beam, under the hypothesis of diagonal constitutive law, and we recognized four different contributions: extensional, shear, flexural and torsional elastic energies. Under reasonable hypotheses of loading conditions, we evaluated the ratios among the four contributions, in terms of characteristic dimensions of the section. We found the following results. (a) *The shear energy is always small compared with the flexural energy*, unless the shear factor is very large. This last case only occurs for homogenized beams, e.g. tower-buildings, in which the (macroscopic) shear strains prevail on the flexural strains. Therefore, referring to an unshearable (or Euler–Bernoulli) beam usually entails a very small error. (b) A similar result was found for the extensional energy. We concluded that *beams can be generally considered as inextensible*, provided that the external constraints permit the ends to approach each other. If, in contrast, such a deformation is prevented, the extensible model must be adopted. (c) Concerning the torsional energy of compact-section or boxed beams, we found that this is small compared with the flexural energy, so that the beams are almost untwistable. If, in contrast, thin-walled beams are considered,

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

torsional and flexural energies are of the same order, and therefore twist must be retained in the model. (d) In the special, but interesting, case of a foil-beam (or *lamina*), the energy related to flexure around the strong axis can be neglected in comparison with the energy for flexure around the weak axis, the latter being of the same order of the torsional energy.

First, we discussed the general strategy, by observing that a scalar formulation is needed, since components of the strain vectors are restrained. We rewrote the VPP in terms of the rates of displacements, $\dot{\boldsymbol{u}}$, $\dot{\boldsymbol{\theta}}$, that we know lead to Lagrangian balance equations. Then, we discussed different methods to approach the problem, namely the mixed, the displacement and the hybrid formulations (Chapter 1). According to the mixed formulation, all the constraints are appended to the VPP by the Lagrange multiplier technique. The multipliers thus assume the meaning of reactive stresses. Balance equations involve both active and reactive stresses. The former can be expressed in terms of displacements, via the constitutive law and kinematics; the latter remain as they are. The problem is therefore governed by the balance equations and the constraints, and appears in mixed displacement-reactive-force form. Alternatively, the displacement method can be used, which requires solving the constraints, in order to express slave variables in terms of master variables. No Lagrange multiplier is therefore used, since the master variables identically satisfy the constraints, and therefore are free variables. The VPP requires two steps: in the first step, an integration by parts is performed to free $\dot{\boldsymbol{u}}$, $\dot{\boldsymbol{\theta}}$ from space-derivatives, by ignoring constraints; in the second step, velocity constraints are introduced in the integral terms and a second integration by parts is performed. As a result, balance equations in the active stresses only are derived. By using constitutive law and kinematics, the equations of motion are finally derived in the master displacements only. Third, a hybrid approach, which combines the features of the two methods, was presented, in which a part of the constraints is solved, and another part is appended to the VPP. After that we browsed several models of beams, with an increasing number of internal constraints. In all cases, we dealt with exact equations.

First, we addressed *unshearable beams*, also called the Euler–Bernoulli beams, in which the cross-sections keep their initial orthogonality to the centerline. The relevant geometrical constraints $\gamma_2 = \gamma_3 = 0$ were first accounted for in the framework of the mixed formulation. Accordingly, the beam is governed by the same balance equations of the unconstrained beam, with the only difference that the dual shear forces T_1, T_2 have a reactive character, and therefore cannot been expressed in terms of strains and, then, displacements. As a result, the final system is mixed in the six displacement fields and two reactive forces; accordingly, the balance equations must be supplemented by the two unshearability conditions. Exact equations were provided. An alternative approach, based on the displacement formulation, was successively followed. Here, the constrains were solved, via algebraic operations only, to express the rotations θ_2, θ_3 as slave of the remaining master variables $\boldsymbol{u}$, θ_1. Time-differentiation of the constraints provided velocity constraints. By using the

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

VPP, four field equations and six alternative boundary conditions were derived. Using the elastic law and the condensed strain–displacement relationships, the field equations and the mechanical boundary conditions were expressed in the master variables only. The whole procedure was restarted for a planar beam, for which explicit expressions of the equations of motion were provided.

As a second example, we added the inextensibility condition $\varepsilon = 0$ to the previous model, thus getting (inextensible) *Euler's elastica*. Since the added constraint only contains space derivatives, it cannot be solved with respect to slave variables without integration. Therefore, we dealt with the inextensibility condition as an auxiliary constraint for the VPP, thus following the hybrid approach. Two different versions of the model were developed: in Version I, the geometrical constraints were taken as they naturally appear; in Version II, they were instead combined, in order to simplify the expressions of the curvatures. The balance equations obtained in the two procedures do not coincide, due to the different meaning of the Lagrange multiplier. The question was studied in more depth referring to a simpler planar model, for which explicit equations were managed. A further version of the planar model was derived, according to the mixed formulation. It was concluded by projection of the vector balance equations in different bases that: (a) in Version I, the Lagrange multipliers assume the meaning of normal force and shear force; (b) in Version II, they correspond to the tangent component and to the component transverse to the axis in the reference configuration; (c) in the mixed formulation, they correspond to the tangent and transverse components to the reference axis.

As a third example, we added the *untwistability condition*. Again, this constraint has to be appended as inextensibility. Thus, we have only two active stresses (the bending moments) and two uncondensed reactive stresses appearing in the balance equations. When the elastic law is used, we have four equations in six mixed unknowns, counterbalanced by the two appended constraints.

The fourth example, concerning the *foil beam*, is similar to the previous example, with a flexure forbidden, but the torsion allowed. Thus, final equations involve the four master displacements and three reactive stresses.

As a fifth example, a model of a *shear–shear–torsional beam* was addressed, representative of a tower building, in which flexure is prevented by the inextensibility of the columns, while shear strains and torsion are allowed by the low flexural rigidity of the columns compared with large flexural stiffness of the (almost rigid) floor-beams. This model was derived according to the displacement formulation. By exploiting the clamped boundary conditions, it was shown that the beam can only experience transverse displacements and twist rotation, all entering the admissible strain–displacement relationships. No appended constraints exist in this special problem, since the constraints are identically satisfied by the vanishing of

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

longitudinal displacement and the two other Tait–Bryan angles. Balance equations in the active T_2, T_3, M_1 stresses were derived, to be put beside the relevant elastic law and kinematics. A brief sketch about identification of the elastic constants was provided. The equation of motion involves the three non-zero displacements only.

All previous models referred to straight beams. As the sixth and final example, we studied a *planar curved beam* in which we introduced the *inextensibility and unshearability* constraints. We illustrated both the hybrid and mixed formulations, thus generalizing the results we achieved for the straight beam.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Chapter 5

Flexible Cables

In this chapter, we develop a model of flexible cable, i.e. of a slender body of negligible flexural and torsional stiffness. First, we carry out an order of magnitude analysis of the elastic potential energy of the Euler–Bernoulli beam, when the slenderness approaches infinity, and provide mathematical arguments for ignoring flexural and torsional contributions. Then, we formulate a model of one-dimensional (1D) Cauchy continuum embedded in a three-dimensional (3D)-space. We start by considering unprestressed cables, i.e. we assume one of the infinite natural states of the cable as reference configuration. Successively, we address prestressed cables, for which the shape assumed by the cable under static loads is taken as the reference configuration. For these cables, we obtain incremental equations of motions, linking the change of stress to the incremental loads. As a particular case, we formulate linearized models. We also specialize all these equations to taut strings, i.e. cables that are rectilinear in the prestressed configuration. Then, we discuss an approximated model for shallow cables, horizontal or inclined. Under quite strong simplifying hypotheses, we condense the tangential motion, and come to a pair of integro-differential equations only in the transverse motion. Finally, we formulate a model for (not shallow) inextensible cables. We retrace the theory for unprestressed and prestressed inextensible cables.

5.1 Flexible cables as a limit of slender beams

The "physical" idea we have of a cable is that of an extremely flexible slender body, which can be bent and twisted with virtually zero force. In bending, we have to enforce small curvature radii, of the order of few times the diameter of the cable,

Mathematical Models of Beams and Cables Angelo Luongo and Daniele Zulli
© 2013 ISTE Ltd. Published 2013 by ISTE Ltd.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

in order to experience a perceptible flexural stiffness; similarly, we have to twist the cable several times, in order to perceive a significant torsional stiffness, namely until the longitudinal fibers dispose themselves in helices whose step is of the order of few times the diameter. In contrast, it is easy to detect a large axial stiffness of the cable. Such experimental observations suggest introducing a mathematical model of a 1D body that has zero bending- and torsion-stiffnesses but a finite axial stiffness. Such an object is *not endowed with its own shape*, since it can undergo infinite *non-rigid* transformations, which require zero deformation work to be spent. In other words, it possesses *infinite natural states*.

To support these qualitative considerations with more rigorous mathematical arguments, we perform an order of magnitude analysis of the quadratic elastic potential of a beam, as we did in Chapter 4 to justify internally constrained models. There, however, we used the *complementary* elastic energy, in order to show that, when order-1 forces are assigned, small energy contributions denote large (at the limit, infinite) stiffness; here, instead, we use the elastic energy, in order to show that, when order-1 displacements are assigned, small energy contributions denote small (at the limit, zero) stiffness. Moreover, we start directly from the Euler–Bernoulli model, for which no shear-strains are allowed, consistent with the fact that we are considering extremely slender beams.

The quadratic, uncoupled, elastic potential energy of a straight beam, deprived of the shear strains, becomes (remember equation [4.1]):

$$\begin{aligned}\phi &= \frac{1}{2}EA\varepsilon^2 + \frac{1}{2}GJ_1\kappa_1^2 + \frac{1}{2}\left(EJ_2\kappa_2^2 + EJ_3\kappa_3^2\right)\\ &= \underbrace{\frac{1}{2}EAu_1'^2}_{=:\phi_e} + \underbrace{\frac{1}{2}GJ_1\theta_1'^2}_{=:\phi_t} + \underbrace{\frac{1}{2}\left(EJ_2u_3''^2 + EJ_3u_2''^2\right)}_{=:\phi_f} + \text{h.o.t.}\end{aligned} \qquad [5.1]$$

where linear kinematics, sufficient to our scope, has been used in writing $\varepsilon \simeq u_1'$, $\kappa_1 \simeq \theta_1'$ and $\kappa_{2,3} \simeq \mp u_{3,2}''$. By accounting for $A = \mathrm{O}\left(r^2\right)$, $J_1 = \mathrm{O}\left(r^4\right)$, $E/G = \mathrm{O}\left(1\right)$, with r as a cross-section characteristic radius, and *assuming that all the displacements vary in space on a characteristic length l of the order of the beam length*, i.e. $u_i' = \mathrm{O}(u_i/l)$, $\theta_1' = \mathrm{O}(\theta_1/l)$, we have:

$$\phi_e = \mathrm{O}\left(E\frac{r^2}{l^2}u_1^2\right), \qquad \phi_t = \mathrm{O}\left(E\frac{r^4}{l^2}\theta_1^2\right), \qquad \phi_f = \mathrm{O}\left(E\frac{r^4}{l^4}u_n^2\right) \qquad [5.2]$$

where $u_n := \left(u_2^2 + u_3^2\right)^{1/2}$ is the modulus of the normal displacement. In order to compare the energy contributions, we have to establish the magnitude of the displacement ratios. A commonly accepted hypothesis is that the tangential displacement u_1 is of the same order, or smaller, than the normal displacement u_n.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Moreover, it can also be assumed that $\theta_1 = O(u_n/l)$, for example of the order $\mathrm{O}\left(10^{-1}\right)$. With these assumptions, it follows that:

$$\frac{\phi_f}{\phi_t} = \mathrm{O}\left(\frac{u_n^2}{l^2\theta_1^2}\right) = \mathrm{O}\left(1\right), \qquad \frac{\phi_f}{\phi_e} = \mathrm{O}\left(\frac{r^2}{l^2}\frac{u_n^2}{u_1^2}\right) = \mathrm{O}\left(\frac{r^2}{l^2}\right) \ll 1 \qquad [5.3]$$

since r/l can be $\mathrm{O}\left(10^{-2}\right)$ or less. In conclusion, while the torsion and flexural energies are comparable, the extension energy is much larger, e.g. $\mathrm{O}\left(10^{4}\right)$ times larger. This result justifies the introduction of an *ideal model* in which the beam possesses just this latter form of energy. Thus, a cable can be viewed as a limit case of a beam in which $J_1, J_2, J_3 \to 0$, i.e. a *(infinitely) flexible and twistable* beam. However, the hypotheses that led us to this result should not be forgotten; namely a cable is flexible when the flexural and torsion curvatures are not too small (i.e. when the characteristic length l is considerably larger than r).

REMARK 5.1. In an infinitely flexible and twistable beam, the couple-stress $\mathbf{m}$ identically goes to zero. For equilibrium reasons, *even the shear forces vanish*, so that the force-stress $\mathbf{t}$ is direct along the tangent to the current configuration (i.e. it possesses only the axial component). Therefore, no reactive force is triggered by the unshearability condition. Later we will see that these properties are consequences of kinematics and the virtual power.

5.2 Unprestressed cables

We develop a model of flexible cable that is stress-free in its reference configuration. Since the attitude of its cross-sections is ineffective in describing the state of the body, we refer to *a 1D Cauchy continuum, immersed in a 3D-space*. Such a continuum (also called not-structured) is made up of points that do *not* posses orientation, so that they are allowed to translate, not to rotate. As a dynamic counterpart, they exchange contact internal forces, not couples.

5.2.1 Kinematics

The reference configuration

Let us consider the cable in any one of the infinite natural configurations, in which it is undeformed and unstressed. We take it as a *reference configuration* assumed at time $t = 0$. To describe it, we use the parametric equation of the centerline $\mathcal{S}$, namely $\bar{\mathbf{x}} = \bar{\mathbf{x}}(s)$, where s is the arclength and $\bar{\mathbf{x}}$ is a vector identifying the position of the generic point P with respect to an arbitrary pole O (Figure 5.1(a)). By taking an external orthogonal basis $\mathcal{B}_e := (\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$, it is $\bar{\mathbf{x}} = \sum_{j=1}^{3} \bar{x}_j(s)\mathbf{i}_j$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

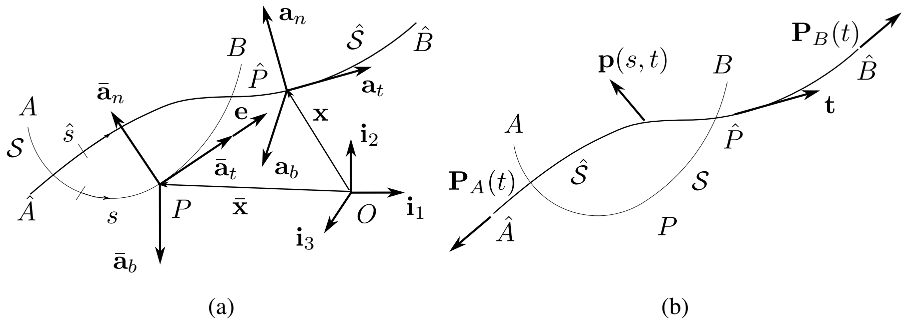


Figure 5.1: Unprestressed cable: (a) kinematics: natural reference configuration $\mathcal{S}$, triad $\bar{\mathcal{B}}_f = (\bar{\mathbf{a}}_t, \bar{\mathbf{a}}_n, \bar{\mathbf{a}}_b)$, arclength s, position vector $\bar{\mathbf{x}}(s)$, strain vector $\mathbf{e}$; current configuration $\hat{\mathcal{S}}$, triad $\mathcal{B}_f = (\mathbf{a}_t, \mathbf{a}_n, \mathbf{a}_b)$, arclength $\hat{s}$, position vector $\mathbf{x}(s)$; external triad $\mathcal{B}_e = (\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$; (b) dynamics: stress vector $\mathbf{t}$, loads $\mathbf{p}, \mathbf{P}_H$.

A Frenet triad $\bar{\mathcal{B}}_f := (\bar{\mathbf{a}}_t(s), \bar{\mathbf{a}}_n(s)\,\bar{\mathbf{a}}_n(s))$ can be built-up on $\mathcal{S}$, similarly to what we did for the curved beam (section 3.1), i.e:

$$\bar{\mathbf{a}}_t = \bar{\mathbf{x}}', \qquad \bar{\mathbf{a}}_n = \frac{1}{\bar{k}}\bar{\mathbf{x}}'', \qquad \bar{\mathbf{a}}_b = \frac{1}{\bar{k}}\left(\bar{\mathbf{x}}' \times \bar{\mathbf{x}}''\right) \qquad [5.4]$$

where:

$$\bar{k} = \|\bar{\mathbf{x}}''\|, \qquad \bar{\tau} = \frac{\bar{\mathbf{x}}' \times \bar{\mathbf{x}}'' \cdot \bar{\mathbf{x}}'''}{\|\bar{\mathbf{x}}''\|^2} \qquad [5.5]$$

are the curvature and the torsion of the line, respectively. The Frenet formulas [2.33] hold for the derivatives of the unit vectors:

$$\bar{\mathbf{a}}_t' = \bar{k}\bar{\mathbf{a}}_n, \qquad \bar{\mathbf{a}}_b' = -\bar{\tau}\bar{\mathbf{a}}_n, \qquad \bar{\mathbf{a}}_n' = \bar{\tau}\bar{\mathbf{a}}_b - \bar{k}\bar{\mathbf{a}}_t \qquad [5.6]$$

REMARK 5.2. We will see that the choice of the natural configuration, to be taken as the reference configuration, does not affect the final results. Moreover, the cable is not requested to satisfy any geometrical boundary conditions in this state. The simplest choice, therefore, would be to take the cable as rectilinear, in such a way $\bar{\mathcal{B}}_f \equiv \mathcal{B}_e$.

REMARK 5.3. The Frenet triad does *not* identify the attitude of the point P, which is not endowed with orientation, but rather the attitude of a *second-order neighborhood* of the point, lying in the osculating plane. As a matter of fact, the first and second derivatives of $\bar{\mathbf{x}}$ appear in the definition of the triad. In other words, while in a polar continuum the position of P is sufficient to identify the local triad, in a non-polar continuum, we also need to know the position of the points close to P. We will return to this point later.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The current configuration

Let the cable assume a (current) configuration at time $t > 0$, in which the centerline lies on the curve $\hat{\mathcal{S}}$ (Figure 5.1). To describe it, we need the parametric equations $\mathbf{x} = \mathbf{x}(s,t)$, or, by components, $\mathbf{x} = \sum_{j=1}^{3} x_j(s,t)\mathbf{i}_j$, which identify the current position $\hat{P}$ of the material point P, which occupied the abscissa s on $\mathcal{S}$. Note that $\hat{\mathcal{S}}$ is *not* parametrized by its "natural" deformed arclength $\hat{s}$, but rather by the undeformed arclength s of the curve $\mathcal{S}$. This entails that the unit vector of the principal basis intrinsic to $\hat{\mathcal{S}}$, as well the Frenet formulas, the curvature and the torsion in the current configuration, all assume more complex expressions. To obtain them, we have to perform the derivatives of composite functions, by taking $\mathbf{x} = \mathbf{x}(s(\hat{s}))$ (time omitted) and applying the chain rule. Thus, e.g., the tangent unit vector is:

$$\mathbf{a}_t := \frac{d\mathbf{x}}{d\hat{s}} = \frac{d\mathbf{x}}{ds}\frac{ds}{d\hat{s}} = \frac{1}{\lambda}\frac{d\mathbf{x}}{ds} \qquad [5.7]$$

where the *stretch*:

$$\lambda := \frac{d\hat{s}}{ds} = \frac{\|d\mathbf{x}\|}{ds} = \|\mathbf{x}'\| \qquad [5.8]$$

has been introduced, equal to the ratio between the deformed and undeformed elemental arcs. By proceeding in a similar way, we obtain the unit vectors of the Frenet triad $\mathcal{B}_f := (\mathbf{a}_t(s,t), \mathbf{a}_n(s,t), \mathbf{a}_b(s,t))$ in the current configuration:

$$\mathbf{a}_t = \frac{1}{\lambda}\mathbf{x}', \qquad \mathbf{a}_n = \frac{1}{\lambda^2 k}\mathbf{x}'', \qquad \mathbf{a}_b = \frac{1}{\lambda^3 k}(\mathbf{x}' \times \mathbf{x}'') \qquad [5.9]$$

and the curvature and torsion:

$$k = \frac{1}{\lambda^2}\|\mathbf{x}''\|, \qquad \tau = \frac{\mathbf{x}' \times \mathbf{x}'' \cdot \mathbf{x}'''}{\lambda^2\|\mathbf{x}''\|^2} \qquad [5.10]$$

Moreover, the following Frenet formulas hold (compare them with equation [2.33], valid for natural parametrization):

$$\mathbf{a}_t' = \lambda k \mathbf{a}_n, \qquad \mathbf{a}_b' = -\lambda\tau\mathbf{a}_n, \qquad \mathbf{a}_n' = \lambda(\tau\mathbf{a}_b - k\mathbf{a}_t) \qquad [5.11]$$

REMARK 5.4. Note that the stretch λ appears in the definition of the current triad of a cable, while no such magnitude appeared in dealing with beams. Indeed, in beams the triad is intrinsic to the point while in cables it is intrinsic to the (stretched) neighborhood.

The geometric boundary conditions

The position vector must satisfy geometric boundary conditions at the constrained ends $H = A, B$, namely:

$$\mathbf{x}_H = \breve{\mathbf{x}}_H \qquad [5.12]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

or:

$$x_{jH} = \breve{x}_{jH}, \qquad j = 1, 2, 3 \qquad [5.13]$$

The macro-rotation

We already noted that the Frenet triads, $\bar{\mathcal{B}}_f, \mathcal{B}_f$, describe the attitude of a neighborhood of a point P. Accordingly, we can define *a rotation of this neighborhood*. To this end, we introduce a tensor $\mathbf{R}$ that leads the triad $\bar{\mathcal{B}}_f$ to match the triad $\mathcal{B}_f$ via:

$$\mathbf{a}_\alpha = \mathbf{R}\bar{\mathbf{a}}_\alpha \qquad \alpha = t, n, b \qquad [5.14]$$

We call $\mathbf{R}$ the *macro-rotation* tensor to be distinguished by the micro-rotation tensor we considered in the polar continuum.

The components of $\mathbf{R}$ in the reference basis can be easily evaluated as:

$$[R_{\alpha\beta}]_{\bar{\mathcal{B}}_f} = \bar{\mathbf{a}}_\alpha \cdot \mathbf{R}\bar{\mathbf{a}}_\beta = \bar{\mathbf{a}}_\alpha \cdot \mathbf{a}_\beta \qquad \alpha, \beta = t, n, b \qquad [5.15]$$

and then expressed, via equations [5.4] and [5.9], in terms of $\bar{\mathbf{x}}$, $\mathbf{x}$ and their derivatives. Therefore, $\mathbf{R}$ is not an independent displacement field. It adds nothing new to the kinematic description, which is completely defined by the position vectors.

REMARK 5.5. Macro- and micro-rotations should not be confused. The former refers to a neighborhood and the latter refers to a point. The former exist both in Cauchy and polar continua and the latter exist only in the polar case[1].

REMARK 5.6. Of course, a displacement field $\mathbf{u} := \mathbf{x} - \bar{\mathbf{x}}$ could be defined, as we did for the beam, measuring the distance between the current and the reference position of a material point. However, this is not of interest for the cable, since, as we said, the reference configuration is arbitrary, and therefore the same current position $\mathbf{x}$ could be identified by infinite $\mathbf{u}$'s. For this reason, we will refer directly to the position $\mathbf{x}$. This choice renders any boundary condition on $\bar{\mathbf{x}}$ meaningless.

The strain vector

In a Cauchy continuum, the strain is only a measure of the non-rigid relative displacements (not of the relative rotations, which are not defined). Therefore,

1. For example, in the linear Timoshenko beam, the shear-strain $\gamma_2 = u_2' - \theta_3$ is the difference between the (infinitesimal) macro-rotation u_2' and the micro-rotation θ_3.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

curvatures have no meaning in cables, and strains reduce to the (reference) strain-vector [2.24], i.e.:

$$\begin{aligned}\mathbf{e} :=& \mathbf{R}^T\mathbf{x}' - \bar{\mathbf{a}}_t = \lambda \mathbf{R}^T \mathbf{a}_t - \bar{\mathbf{a}}_t \\ &= (\lambda - 1)\, \bar{\mathbf{a}}_t\end{aligned} \quad [5.16]$$

where we wrote $\bar{\mathbf{a}}_t$ instead of $\bar{\mathbf{a}}_1$, used equation [5.9a], and finally equation [5.14]. Hence (as we already observed for the unshearable beam (Chapter 4)), the strain vector is tangent to the centerline. By representing it as:

$$\mathbf{e} := e\bar{\mathbf{a}}_t \quad [5.17]$$

it follows that the unique scalar strain component is $e = \lambda - 1$, which represents the *unit extension*[2]; in terms of the position vector, we have:

$$e = \|\mathbf{x}'\| - 1 \quad [5.18]$$

By using the components of $\mathbf{x}$ in the external basis, the strain becomes:

$$e = \sqrt{x_1'^2 + x_2'^2 + x_3'^2} - 1 \quad [5.19]$$

The macro-spin

The velocity of a cable is completely described by the vector field $\mathbf{v} := \dot{\mathbf{x}}\,(s,t)$. However, as we did for the rotation, we could be interested in determining the *spin* of a neighborhood of the generic point P, that we will call the *macro-spin*. To evaluate it, we remember (equation [2.76]) that, in a rigid motion, the velocity gradient is $\mathbf{v}' = \boldsymbol{\omega} \times \mathbf{x}'$. This formula can be inverted by vector-multiplication of both members by $\mathbf{x}'$, by additionally requiring that $\boldsymbol{\omega} \perp \mathbf{x}'$ (since the component of $\boldsymbol{\omega}$ parallel to $\mathbf{x}'$ does not contribute to $\mathbf{v}'$). It follows that[3]:

$$\boldsymbol{\omega} = \frac{1}{\lambda^2}\,(\mathbf{x}' \times \mathbf{v}') \quad [5.20]$$

We take this expression as the definition of the macro-spin. Indeed, if $\mathbf{v}'$ is rigid, it returns the spin that produced the rigid gradient; if $\mathbf{v}'$ is non-rigid, the formula gives the spin that extracts its rigid part from the gradient.

---

2. We use the symbol $e = \|\mathbf{e}\|$ to denote the unit extension, although it coincides with the longitudinal strain ε we used for the unshearable beam.

3. By expanding the double cross-product, we have:

$$\mathbf{x}' \times \mathbf{v}' = \mathbf{x}' \times (\boldsymbol{\omega} \times \mathbf{x}') = (\mathbf{x}' \cdot \mathbf{x}')\,\boldsymbol{\omega} - \underbrace{(\mathbf{x}' \cdot \boldsymbol{\omega})}_{=0}\mathbf{x}'$$

from which, by using equations [5.8] and [5.20], follows.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The stretching velocity gradient

By closely following section 2.1.8, we introduce a (material) velocity gradient $\mathbf{g} := \mathbf{v}'$; subtracting from it the rigid gradient $\mathbf{w} = \boldsymbol{\omega} \times \mathbf{x}'$, in which $\boldsymbol{\omega}$ is the macro-spin [5.20], we obtain the stretching velocity gradient $\mathbf{d} := \mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}'$. By transforming this expression, we have:

$$\begin{aligned} \mathbf{d} :=& \mathbf{v}' + \frac{\mathbf{x}' \times (\mathbf{x}' \times \mathbf{v}')}{\lambda^2} \\ =& \mathbf{v}' + \frac{(\mathbf{x}' \cdot \mathbf{v}')\, \mathbf{x}' - (\mathbf{x}' \cdot \mathbf{x}')\, \mathbf{v}'}{\lambda^2} \\ =& (\mathbf{v}' \cdot \mathbf{a}_t)\, \mathbf{a}_t \end{aligned} \qquad [5.21]$$

where, in the order, we substituted equation [5.20], expanded the double cross-product and used $\mathbf{x}' = \lambda \mathbf{a}_t$ (equation [5.9a]). Remarkably, we find that *the stretching velocity gradient is the component of the velocity gradient* $\mathbf{v}'$ *tangent to the centerline in the current configuration*. Therefore, the transverse component is responsible for the macro-spin.

REMARK 5.7. The decomposition of the velocity gradient *vector* in the tangent (stretching) and normal (spin effect) components is the counterpart of what occurs in the 3D Cauchy continuum, in which the velocity gradient *tensor* decomposes in its symmetric (stretching) and skew-symmetric (spin) parts.

The strain rate

First, we define the (scalar) strain-rate as the time-derivative of the strain [5.18], namely[4]:

$$\dot{e} = \frac{\partial \sqrt{\mathbf{x}' \cdot \mathbf{x}'}}{\partial t} = \frac{\dot{\mathbf{x}}' \cdot \mathbf{x}'}{\|\mathbf{x}'\|} = \frac{\mathbf{v}' \cdot \mathbf{x}'}{\lambda} = \mathbf{v}' \cdot \mathbf{a}_t \qquad [5.22]$$

in which we used $\mathbf{x}' = \lambda \mathbf{a}_t$. Consequently, the vector strain-rate becomes:

$$\dot{\mathbf{e}} = \dot{e}\bar{\mathbf{a}}_t = (\mathbf{v}' \cdot \mathbf{a}_t)\, \bar{\mathbf{a}}_t \qquad [5.23]$$

4. In the symbolism of the metamodel (Chapter 1), since $\mathbf{a}_t = (1/\lambda) \sum_{j=1}^{3} x_j' \mathbf{i}_j$, we have:

$$\dot{e} = \frac{1}{\lambda} \begin{bmatrix} x_1' \partial_s & x_2' \partial_s & x_3' \partial_s \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$$

which defines the *kinematic operator* $\boldsymbol{D}$ as a 3×1 (formal) matrix. This circumstance, once again, shows that the flexible cable is a *locally undetermined mechanical system*, since there exist non-rigid velocity fields for which $\dot{e} = 0$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

If we compare the stretching velocity gradient [5.21] with the strain-rate so far obtained, we find:

$$\mathbf{d} = \mathbf{R}\dot{\mathbf{e}} \qquad [5.24]$$

i.e. the same result holding for the beam (equation [2.79]), namely the stretching velocity gradient is equal to the strain-rate vector pushed-forward.

5.2.2 Dynamics

Now, we address the dynamic aspects of the model. Let us consider the cable loaded by external forces of linear density $\mathbf{p}(s,t)$ acting in the domain (having the meaning of force per unit of *undeformed* length), and by forces $\mathbf{P}_H(t)$ applied at the boundaries $H = A, B$ (see Figure 5.1(b)). No couples can be applied, since the continuum is non-polar. To obtain the equations governing the dynamics, we first execute the balance of powers, then, as an alternative, the balance of forces.

The Virtual Power Principle

By superimposing a virtual motion $\mathbf{v}$ to the current configuration, we write the expressions for the external and internal virtual powers:

$$\begin{aligned} \mathcal{P}_{ext} &:= \int_{\mathcal{S}} \mathbf{p} \cdot \mathbf{v} ds + \sum_{H=A}^{B} \mathbf{P}_H \cdot \mathbf{v}_H \\ \mathcal{P}_{int} &:= \int_{\mathcal{S}} \mathbf{t} \cdot \mathbf{d} ds \end{aligned} \qquad [5.25]$$

Here, $\mathbf{t}$ is a force-stress that spends power in the stretching velocity gradient $\mathbf{d}$. Since the latter, by equation [5.21], is collinear to $\mathbf{a}_t$, the stress $\mathbf{t}$ must also be collinear to $\mathbf{a}_t$, i.e.:

$$\mathbf{t} = T\mathbf{a}_t \qquad [5.26]$$

where T is referred to as the *tension* of the cable[5]. This is a remarkable result that *the cable is only capable to provide axial forces*. It should be noticed that this is *not* an additional hypothesis, but a consequence of kinematics and Virtual Power Principle.

5. We use the symbol $T = \|\mathbf{t}\|$ to denote the tension, although it coincides with the normal force N we used for the beam. However, while the axial force is a *component* of $\mathbf{t}$, since shear forces do exist in a beam, the tension is the *modulus* of $\mathbf{t}$, since no shear forces are present in a cable.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By using equation [5.21], the Virtual Power Principle (VPP) becomes[6]:

$$\int_{\mathcal{S}} \mathbf{p} \cdot \mathbf{v} ds + \sum_{H=A}^{B} \mathbf{P}_H \cdot \mathbf{v}_H = \int_{\mathcal{S}} \mathbf{t} \cdot \mathbf{v}' ds \qquad \forall \mathbf{v} \tag{5.27}$$

or, after integration by parts:

$$\int_{\mathcal{S}} (\mathbf{t}' + \mathbf{p}) \cdot \mathbf{v} ds + \sum_{H=A}^{B} [(\mathbf{P}_H \pm \mathbf{t}_H) \cdot \mathbf{v}_H] = 0 \qquad \forall \mathbf{v} \tag{5.28}$$

This leads to the following local balance equation:

$$\mathbf{t}' + \mathbf{p} = \mathbf{0} \tag{5.29}$$

and the alternate boundary conditions:

$$(\mathbf{P}_H \pm \mathbf{t}_H) \cdot \mathbf{v}_H = 0 \tag{5.30}$$

Inertial effects can be accounted via the d'Alembert principle for which the field equation modifies into:

$$\mathbf{t}' + \mathbf{p} = m\ddot{\mathbf{x}} \tag{5.31}$$

REMARK 5.8. The balance equations are well-known, since they formally coincide with *a part* of the equations we obtained for the beam (equation [2.103]); however, it should be kept in mind that while $\mathbf{t}$ can be any vector of the space for the beam (i.e. it has three non-zero components), $\mathbf{t}$ must be parallel to $\mathbf{a}_t$ for the cable (i.e. it has just one component).

REMARK 5.9. Since, by virtue of equation [5.24], $\mathbf{t} \cdot \mathbf{d} = \mathbf{t} \cdot \mathbf{R}\dot{\mathbf{e}}$, and, moreover, $\mathbf{t} = T\mathbf{a}_t$ and $\dot{\mathbf{e}} = \dot{e}\bar{\mathbf{a}}_t$, the internal virtual power can also be written in terms of strain-rate component:

$$\mathcal{P}_{int} = \int_{\mathcal{S}} T\dot{e} ds \tag{5.32}$$

which is a particular case of equation [2.98], valid for the beam.

6. Indeed $\mathbf{t} \cdot \mathbf{d} = (\mathbf{v}' \cdot \mathbf{a}_t)(\mathbf{t} \cdot \mathbf{a}_t) = \mathbf{v}' \cdot \mathbf{a}_t T = \mathbf{t} \cdot \mathbf{v}'$. Alternatively:

$$\mathbf{t} \cdot \mathbf{d} = \mathbf{t} \cdot (\mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}') = \mathbf{t} \cdot \mathbf{v}' - \mathbf{x}' \times \mathbf{t} \cdot \boldsymbol{\omega} = \mathbf{t} \cdot \mathbf{v}'$$

since $\mathbf{t} \parallel \mathbf{x}'$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The momentum principles

We could ask ourselves the reason for which *the dynamics of the flexible cable is only governed by the linear momentum equation*, while the angular momentum equation disappears. The answer is given by the alternative approach based on the balance of the forces.

The linear momentum principle, written for a segment of cable, formally reads as for the beam (see equation [2.125]), and therefore leads to equation [5.31]. The angular momentum principle, instead, still reads as in equation [2.128], but deprived of the (internal and external) couple contributions, as well as of the rotatory part of the angular momentum. Thus, in the localized balance equation [2.131], only terms of non-polar type survive, namely:

$$\mathbf{x}' \times \mathbf{t} = \mathbf{0} \qquad [5.33]$$

i.e. an equation *which is trivially satisfied*, since $\mathbf{t} \| \mathbf{x}'$.

Concerning boundary conditions of a mechanical type, since no external couples are applied and no internal couples can emerge at the boundaries, no further requirements must be made, in addition to equation [5.30].

REMARK 5.10. The absence of a moment condition for the flexible cable depends on the fact that, since the stress-force is direct along the tangent to the centerline, and since no couples can act on the cable, rotational equilibrium of an infinitesimal segment is always satisfied. This circumstance is similar to that met in the 3D Cauchy continuum, where the symmetry of the stress tensor assures the rotational equilibrium of an infinitesimal parallelepiped element.

The scalar balance equations in the external basis

The balance equations [5.29] and [5.30] are projected onto the external basis $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$. By letting:

$$\mathbf{p} := \sum_{j=1}^{3} p_j \mathbf{i}_j, \qquad \mathbf{P}_H := \sum_{j=1}^{3} P_{jH} \mathbf{i}_j \qquad [5.34]$$

and accounting for:

$$\mathbf{t} = T\mathbf{a}_t = T\frac{\mathbf{x}'}{\lambda} = \frac{T}{1+e}\sum_{j=1}^{3} x_j' \mathbf{i}_j \qquad [5.35]$$

the scalar field equations follow:

$$\begin{aligned}&\frac{\partial}{\partial s}\left(\frac{T}{1+e}\frac{\partial x_1}{\partial s}\right)+p_1=m\ddot{x}_1\\&\frac{\partial}{\partial s}\left(\frac{T}{1+e}\frac{\partial x_2}{\partial s}\right)+p_2=m\ddot{x}_2\\&\frac{\partial}{\partial s}\left(\frac{T}{1+e}\frac{\partial x_3}{\partial s}\right)+p_3=m\ddot{x}_3\end{aligned} \qquad [5.36]$$

with the relevant boundary conditions:

$$\mp\left[\frac{T}{1+e}\frac{\partial x_j}{\partial s}\right]_H=P_{jH}, \qquad j=1,2,3 \qquad [5.37]$$

In statics, the field equations reduce to:

$$\begin{aligned}&\frac{d}{ds}\left(\frac{T}{1+e}\frac{dx_1}{ds}\right)+p_1=0\\&\frac{d}{ds}\left(\frac{T}{1+e}\frac{dx_2}{ds}\right)+p_2=0\\&\frac{d}{ds}\left(\frac{T}{1+e}\frac{dx_3}{ds}\right)+p_3=0\end{aligned} \qquad [5.38]$$

REMARK 5.11. The field and boundary conditions can be easily interpreted, if we consider that $T\partial x_j/\partial\hat{s}$ represents the projection of the internal force on the jth unit vector of the external basis.

The static equations in the current basis

The projection of the balance equations [5.29] and [5.30] onto the (unknown) current basis $\mathcal{B}_f$ is not convenient in the dynamic case[7]. Nevertheless, in the static case, the equations considerably simplify and their detection provides insights into the mechanical behavior of the cable.

The internal force, in the current basis, becomes:

$$\mathbf{t}'=(T\mathbf{a}_t)'=T'\mathbf{a}_t+T\lambda k\mathbf{a}_n \qquad [5.39]$$

7. Indeed, the inertia forces assume the cumbersome form:

$$\begin{pmatrix}p_t^{in}\\p_n^{in}\\p_b^{in}\end{pmatrix}:=-m\begin{pmatrix}\mathbf{a}_t\cdot\mathbf{i}_1&\mathbf{a}_t\cdot\mathbf{i}_2&\mathbf{a}_t\cdot\mathbf{i}_3\\\mathbf{a}_n\cdot\mathbf{i}_1&\mathbf{a}_n\cdot\mathbf{i}_2&\mathbf{a}_n\cdot\mathbf{i}_2\\\mathbf{a}_b\cdot\mathbf{i}_1&\mathbf{a}_b\cdot\mathbf{i}_2&\mathbf{a}_b\cdot\mathbf{i}_2\end{pmatrix}\begin{pmatrix}\ddot{x}_1\\\ddot{x}_2\\\ddot{x}_3\end{pmatrix}$$

where the elements of the matrix must be evaluated via equation [5.9].

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

in which we used $\mathbf{a}'_t = \lambda k \mathbf{a}_n$ (equation [5.11a]); with regard to external forces, we have:

$$\begin{aligned} \mathbf{p} &:= p_t \mathbf{a}_t + p_n \mathbf{a}_n + p_b \mathbf{a}_b \\ \mathbf{P}_H &= [P_t \mathbf{a}_t + P_n \mathbf{a}_n + P_b \mathbf{a}_b]_H \end{aligned} \tag{5.40}$$

Thus, we obtain the static field equations:

$$\begin{aligned} &T' + p_t = 0 \\ &\lambda T k + p_n = 0 \\ &p_b = 0 \end{aligned} \tag{5.41}$$

sided by the following boundary conditions:

$$\mp T_H = P_{tH}, \qquad 0 = P_{nH}, \qquad 0 = P_{bH} \tag{5.42}$$

The scalar field equation [5.41c] states that $p_b = 0$, i.e. $\mathbf{p} = p_t \mathbf{a}_t + p_n \mathbf{a}_n$ (remember that $\mathbf{a}_t, \mathbf{a}_n$ are unknowns!). This circumstance has a strong mechanical meaning, namely the cable locally disposes itself in such a way that *the osculating plane contains the force density* $\mathbf{p}$. Similarly, the mechanical boundary conditions [5.42] state that, at the free boundary, the tangent at the centerline $\mathbf{a}_{tH}$ is aligned with the prescribed force $\mathbf{P}_H$, if this is different from zero. If, in contrast, the force is zero, then $\mathbf{t}_H$ goes to zero and the tangent is arbitrary.

A remarkable case occurs when $p_t \equiv 0$, i.e. when $\mathbf{p} = p_n \mathbf{a}_n$; this happens, e.g., when the cable is loaded by *pressure forces*, which, for their nature, are normal to the surface on which they act. In this case, equation [5.41a] provides $T =$ const and equation [5.41b] provides:

$$k = -\frac{\hat{p}_n}{T} \tag{5.43}$$

where $\hat{p} := -p_n/\lambda$. This quantity has a clear meaning emerging from the force continuity law $p_n ds = \hat{p} d\hat{s}$; it is therefore *the force per unit of stretched length* of the cable. If also $\hat{p} =$ const, then the cable disposes itself along a circumference of radius $R := \left|k^{-1}\right| = |T/\hat{p}_n|$, as it is well-known from elementary mechanics. The minus sign denotes that concavity is opposite to the pressure.

5.2.3 Constitutive law

We derive constitutive laws by specializing those relevant to a beam.

For *hyperelastic* material, the Green law [2.151] reduces to:

$$\mathbf{R}^T \mathbf{t} = \frac{\partial \phi(\mathbf{e})}{\partial \mathbf{e}} \tag{5.44}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

It states that *the pulled-back stress,* $\mathbf{R}^T\mathbf{t}$*, equates the derivative of the elastic potential* ϕ *with respect to the strain* $\mathbf{e}$. In scalar form, since $\mathbf{R}^T\mathbf{t} = T\bar{\mathbf{a}}_t$ and $\frac{\partial\phi}{\partial\mathbf{e}} = \frac{\partial\phi}{\partial e}\bar{\mathbf{a}}_t$, we have:

$$T = \frac{\partial\phi}{\partial e} \qquad [5.45]$$

If $\phi = \frac{1}{2}EAe^2$ is taken, then:

$$T = EAe \qquad [5.46]$$

where EA is the axial stiffness of the flexible cable.

For linear viscoelastic materials, if we adopt the Kelvin–Voigt model (see equation [2.180]), we have:

$$T = EAe + \eta A\dot{e} \qquad [5.47]$$

where η is the viscosity coefficient. According to the standard model, we instead have (see equation [2.181]):

$$\dot{T} + \frac{E_0 + E_v}{\eta}T = E_0A\dot{e} + \frac{E_0E_v}{\eta}Ae \qquad [5.48]$$

where E_0, E_v are two elastic moduli.

5.2.4 The Fundamental Problem

The Fundamental Problem for the flexible cable is governed by the following set of equations:

– One strain-position scalar relationship [5.19]:

$$e = \sqrt{x_1'^2 + x_2'^2 + x_3'^2} - 1 \qquad [5.49]$$

– Three balance scalar equations that, in the external basis, become (equation [5.36]):

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(\frac{T}{1+e}\frac{\partial x_1}{\partial s}\right) + p_1 = m\ddot{x}_1\\
&\frac{\partial}{\partial s}\left(\frac{T}{1+e}\frac{\partial x_2}{\partial s}\right) + p_2 = m\ddot{x}_2\\
&\frac{\partial}{\partial s}\left(\frac{T}{1+e}\frac{\partial x_3}{\partial s}\right) + p_3 = m\ddot{x}_3
\end{aligned} \qquad [5.50]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

– One constitutive equation, which is equation [5.46] if the material is elastic:

$$T = EAe \qquad [5.51]$$

– Alternative geometrical/mechanical boundary conditions (equations [5.61] and [5.37]):

$$\begin{aligned} & x_{jH} = \breve{x}_{jH}, \qquad j = 1, 2, 3 \\ & \mp \left[\frac{T}{1+e} \frac{\partial x_j}{\partial s} \right]_H = P_{jH}, \qquad j = 1, 2, 3 \end{aligned} \qquad [5.52]$$

The unknowns of the problem are the components of the position vector x_j, the unit extension e, and the stress T. Overall, five differential/algebraic equations in five unknowns. By formulating the problem only in terms of the position, we finally have:

$$\begin{aligned} & \frac{\partial}{\partial s} \left(\frac{EAe}{1+e} \frac{\partial x_1}{\partial s} \right) + p_1 = m\ddot{x}_1 \\ & \frac{\partial}{\partial s} \left(\frac{EAe}{1+e} \frac{\partial x_2}{\partial s} \right) + p_2 = m\ddot{x}_2 \\ & \frac{\partial}{\partial s} \left(\frac{EAe}{1+e} \frac{\partial x_3}{\partial s} \right) + p_3 = m\ddot{x}_3 \end{aligned} \qquad [5.53]$$

with the mechanical boundary conditions:

$$\mp \left[\frac{EAe}{1+e} \frac{\partial x_j}{\partial s} \right]_H = P_{jH}, \qquad j = 1, 2, 3 \qquad [5.54]$$

where $e = \|\mathbf{x}'\| - 1$. Usually $e \ll 1$, so that $1 + e \simeq 1$ could be taken, to slightly simplify the previous equations.

The planar force case

When the external forces are planar, e.g. contained in the $(\mathbf{i}_1, \mathbf{i}_2)$-plane, the current configuration of the cable is also contained in the plane (consistently with what was already observed about the fact that the osculating plane contains the force). As a matter of fact, since $p_3 = 0$, $P_{3H} = 0$, equations [5.36] and [5.37] admit the solution $x_3 = 0$, and they reduce to:

$$\begin{aligned} & \frac{\partial}{\partial s} \left(\frac{T}{1+e} \frac{\partial x_1}{\partial s} \right) + p_1 = m\ddot{x}_1 \\ & \frac{\partial}{\partial s} \left(\frac{T}{1+e} \frac{\partial x_2}{\partial s} \right) + p_2 = m\ddot{x}_2 \end{aligned} \qquad [5.55]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and:

$$\mp\left[\frac{T}{1+e}\frac{\partial x_1}{\partial s}\right]_H = P_{1H}, \qquad \mp\left[\frac{T}{1+e}\frac{\partial x_2}{\partial s}\right]_H = P_{2H} \qquad [5.56]$$

The unit strain [5.19] simplifies into:

$$e = \sqrt{x_1'^2 + x_2'^2} - 1 \qquad [5.57]$$

Geometric boundary conditions, if any, should be compatible with the planar solution, i.e. they must prescribe that the end points(s) belong to the same plane:

$$x_{1H} = \breve{x}_{1H}, \qquad x_{2H} = \breve{x}_{2H}, \qquad x_{3H} = 0 \qquad [5.58]$$

5.3 Prestressed cables

In the previous section, we formulated a mathematical model of flexible cable that, in the reference configuration, is stress-free. However, very often, we are interested in evaluating the response of cables that are *prestressed by static force*s. A typical case is represented by cables in equilibrium under their own weight (as the electrical lines), or permanent loads (as suspended bridges), solicited by incremental forces, such as wind- or traffic-loads. In these cases, it is convenient just to take the prestressed configuration as reference configuration, and to refer all quantities to it. In particular, using relative positions (i.e. displacements), instead of absolute positions, allows some simplifications, as series expansions around equilibrium or even linearization. For example, we could be interested in oscillations of small amplitude around the equilibrium configuration, for which the linearized theory is sufficient to the purpose.

According to these ideas, we want to formulate a model of prestressed cable. We assume that a prestress analysis, able to describe the transformation of the cable from the natural to the reference configuration, has already been performed by using the tools described in the previous sections, so that we will focus our attention on the study of the response of the cable to *incremental loads*, of static or dynamic nature. We will mainly use exact equations, except for assuming $\lambda = 1$ in the balance equations; for this reason, will call the model *quasi-exact*. Then, we will linearize it.

5.3.1 Quasi-exact models

Prestressed reference state

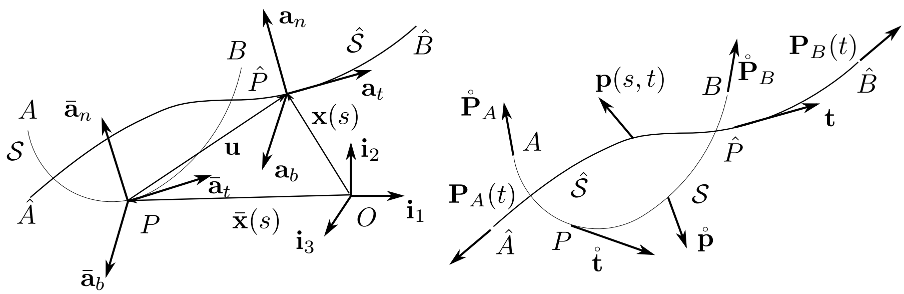
Let us consider a flexible cable, lying on the curve $\mathcal{S}$ of the 3D-space, described by the (natural) parametric equations $\bar{\mathbf{x}} = \bar{\mathbf{x}}(s)$, where s is the arclength (Figure 5.2a).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By denoting by $\bar{\mathcal{B}}_f := (\bar{\mathbf{a}}_t(s), \bar{\mathbf{a}}_n(s), \bar{\mathbf{a}}_n(s))$ the relevant Frenet triad, the formulas [5.6] hold. Let the cable be in equilibrium under static external loads $\mathring{\mathbf{p}}(s), \mathring{\mathbf{P}}_H$ and internal stress $\mathring{\mathbf{t}}(s)$, so that the balance equations [5.69] and the boundary conditions [5.30] are satisfied, i.e.:

$$\begin{aligned} &\mathring{\mathbf{t}}' + \mathring{\mathbf{p}} = \mathbf{0} \\ &\mp \mathring{\mathbf{t}}_H = \mathring{\mathbf{P}}_H \end{aligned} \qquad [5.59]$$

REMARK 5.12. It should be noted that s is *a stretched abscissa*, as a result of the transformation undergone by the cable in passing from its natural to the prestressed configuration. However, according to the referential description, we can forget such a transformation, and take s as the material abscissa.



(a) (b)

Figure 5.2: Prestressed cable: (a) kinematics: reference prestressed configuration $\mathcal{S}$, current configuration $\hat{\mathcal{S}}$, displacement $\mathbf{u}$; (b) dynamics: preloads $\mathring{\mathbf{p}}, \mathring{\mathbf{P}}_H$, prestress $\mathring{\mathbf{t}}$, current loads $\mathbf{p}, \mathbf{P}_H$, current stress $\mathbf{t}$.

Kinematics

Let the cable occupy, at time t, a current configuration described by the (non-natural) parametric equations $\mathbf{x} = \mathbf{x}(s,t)$. Let $\mathcal{B}_f := (\mathbf{a}_t(s,t), \mathbf{a}_n(s,t), \mathbf{a}_b(s,t))$ be the relevant principal triad, whose vectors satisfy the Frenet formulas [5.11]. We define the displacement vector:

$$\mathbf{u} := \mathbf{x}(s,t) - \bar{\mathbf{x}}(s) \qquad [5.60]$$

and describe the current configuration in terms of it. Displacements are restrained by geometric boundary conditions at the constrained ends:

$$\mathbf{u}_H = \breve{\mathbf{u}}_H \qquad [5.61]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

We assume the unit strain [5.18] as a (scalar) measure of the strain; by using equation [5.60], and remembering that $\bar{\mathbf{x}}' = \bar{\mathbf{a}}_t$, it becomes:

$$e = \sqrt{1 + 2\mathbf{u}' \cdot \bar{\mathbf{a}}_t + \mathbf{u}' \cdot \mathbf{u}'} - 1 \qquad [5.62]$$

By representing the displacement in the external basis, we have:

$$\mathbf{u} := \sum_{j=1}^{3} u_j \mathbf{i}_j \qquad [5.63]$$

Since $\bar{\mathbf{a}}_t = \sum_{j=1}^{3} \bar{x}'_j \mathbf{i}_j$, then:

$$e = \sqrt{1 + 2\left(\sum_{j=1}^{3} \bar{x}'_j u'_j + \frac{1}{2}\sum_{j=1}^{3} u'^2_j\right)} - 1 \qquad [5.64]$$

If, in contrast, we represent the displacement in the intrinsic reference basis, i.e. if we put:

$$\mathbf{u} := u_t \bar{\mathbf{a}}_t + u_n \bar{\mathbf{a}}_n + u_b \bar{\mathbf{a}}_b \qquad [5.65]$$

and, moreover, we use the Frenet formulas [5.11] to express $\mathbf{u}'$ (or, equivalently, the Poisson formula in $\bar{\mathcal{B}}_f$), we have:

$$\mathbf{u}' = \left(u'_t - \bar{k} u_n\right) \bar{\mathbf{a}}_t + \left(u'_n + \bar{k} u_t - \bar{\tau} u_b\right) \bar{\mathbf{a}}_n + \left(u'_b + \bar{\tau} u_n\right) \bar{\mathbf{a}}_b \qquad [5.66]$$

Consequently, we obtain:

$$\begin{aligned} e = \Big\{ & 1 + 2\left(u'_t - \bar{k} u_n\right) \\ & + \left[\left(u'_t - \bar{k} u_n\right)^2 + \left(u'_n + \bar{k} u_t - \bar{\tau} u_b\right)^2 + \left(u'_b + \bar{\tau} u_n\right)^2\right]\Big\}^{1/2} - 1 \end{aligned} \qquad [5.67]$$

For later purposes, we also evaluate the second derivative of $\mathbf{u}$:

$$\begin{aligned} \mathbf{u}'' = & \left[\left(u'_t - \bar{k} u_n\right)' - \bar{k}\left(u'_n + \bar{k} u_t - \bar{\tau} u_b\right)\right] \bar{\mathbf{a}}_t \\ & + \left[\left(u'_n + \bar{k} u_t - \bar{\tau} u_b\right)' + \bar{k}\left(u'_t - \bar{k} u_n\right) - \bar{\tau}\left(u'_b + \bar{\tau} u_n\right)\right] \bar{\mathbf{a}}_n \\ & + \left[\left(u'_b + \bar{\tau} u_n\right)' + \bar{\tau}\left(u'_n + \bar{k} u_t - \bar{\tau} u_b\right)\right] \bar{\mathbf{a}}_b \end{aligned} \qquad [5.68]$$

The incremental balance equations

The balance equations are given by equations [5.31] and [5.30] that, in terms of displacement, become:

$$\begin{aligned} & \mathbf{t}' + \mathbf{p} = m\ddot{\mathbf{u}} \\ & \mp \mathbf{t}_H = \mathbf{P}_H \end{aligned} \qquad [5.69]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Here, $\mathbf{t}(s,t)$ and $\mathbf{p}(s,t)$ represent the stress and the force acting in the current configuration. In order to express the equations in terms of increments with respect to the reference configuration, we subtract equation [5.59] from the previous equations, by obtaining:

$$\begin{aligned} &\left(\mathbf{t}-\mathring{\mathbf{t}}\right)' + \tilde{\mathbf{p}} = m\ddot{\mathbf{u}} \\ \mp &\left(\mathbf{t}-\mathring{\mathbf{t}}\right) = \tilde{\mathbf{P}}_H \end{aligned} \qquad [5.70]$$

where:

$$\tilde{\mathbf{p}} := \mathbf{p} - \mathring{\mathbf{p}}, \qquad \tilde{\mathbf{P}}_H := \mathbf{P}_H - \mathring{\mathbf{P}}_H \qquad [5.71]$$

are *incremental loads*. Equations [5.70] are called the *incremental balance equations*.

To express them in scalar form, first we have to note that $\mathbf{t}$ and $\mathring{\mathbf{t}}$ *are generally non-parallel*, since $\mathbf{t} = T\mathbf{a}_t$ and $\mathring{\mathbf{t}} = \mathring{T}\bar{\mathbf{a}}_t$. Since, from equations [5.9a] and [5.60], it is $\mathbf{a}_t = \left(\bar{\mathbf{a}}_t + \mathbf{u}'\right)/\lambda$, then:

$$\mathbf{t} - \mathring{\mathbf{t}} = \frac{T}{\lambda}\left(\bar{\mathbf{a}}_t + \mathbf{u}'\right) - \mathring{T}\bar{\mathbf{a}}_t \qquad [5.72]$$

It appears clearly that *the incremental balance equations provide a significant simplification if and only if we can assume* $\lambda = 1$[8]. The hypothesis seems acceptable in the engineering field, where usually $\lambda = 1 + \mathrm{O}\left(10^{-3}\right)$; for this reason, we will use it and call the relevant model *quasi-exact*. Accordingly, equations [5.70] become:

$$\begin{aligned} &\left[\tilde{T}\bar{\mathbf{a}}_t + T\mathbf{u}'\right]' + \tilde{\mathbf{p}} = m\ddot{\mathbf{u}} \\ \mp &\left[\tilde{T}\bar{\mathbf{a}}_t + T\mathbf{u}'\right]_H = \tilde{\mathbf{P}}_H \end{aligned} \qquad [5.73]$$

where $\tilde{T} := T - \mathring{T}$ is the increment of (scalar) stress. When these equations are projected onto the external basis, they furnish:

$$\left[\tilde{T}\bar{x}_j' + Tu_j'\right]' + \tilde{p}_j = m\ddot{u}_j, \qquad j = 1,2,3 \qquad [5.74]$$

and:

$$\mp\left[\tilde{T}\bar{x}_j' + Tu_j'\right]_H = \tilde{P}_{jH}, \qquad j = 1,2,3 \qquad [5.75]$$

8. Indeed, if this is not the case, it is preferable to use the current balance equations [5.69]

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

If equations [5.73] are projected onto the intrinsic reference basis, with the help of equations [5.66] and [5.68], they provide:

$$\begin{aligned}
&\tilde{T}' + \left[T\left(u_t' - \bar{k}u_n\right)\right]' - \bar{k}T\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right) + \tilde{p}_t = m\ddot{u}_t \\
&\bar{k}\tilde{T} + \left[T\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right]' \\
&+ \bar{k}T\left(u_t' - \bar{k}u_n\right) - \bar{\tau}\left(u_b' + \bar{\tau}u_n\right) + \tilde{p}_n = m\ddot{u}_n \\
&\left[T\left(u_b' + \bar{\tau}u_n\right)\right]' + \bar{\tau}T\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right) + \tilde{p}_b = m\ddot{u}_b
\end{aligned} \qquad [5.76]$$

and:

$$\begin{aligned}
&\mp\left[\tilde{T} + T\left(u_t' - \bar{k}u_n\right)\right]_H = \tilde{P}_t \\
&\mp\left[T\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right]_H = \tilde{P}_n \\
&\mp\left[T\left(u_b' + \bar{\tau}u_n\right)\right]_H = \tilde{P}_b
\end{aligned} \qquad [5.77]$$

The elastic law

We confine ourselves to the a hyperelastic material. By taking a quadratic non-homogeneous potential, $\phi = \mathring{T}e + \frac{1}{2}EAe^2$, and using the Green law $T = \frac{\partial\phi}{\partial e}$, we derive:

$$T = \mathring{T} + EAe \qquad [5.78]$$

where EA is the axial stiffness of the flexible cables

REMARK 5.13. Note that $EA = \left(\frac{\partial^2\phi}{\partial e^2}\right)_{e=0}$. However, $e = 0$ now identifies the prestressed configuration, while in the previous section it refers to the natural configuration. Therefore, strictly speaking, the axial stiffness of prestressed cables is different from that of unprestressed cables. Usually, however, such a difference is neglected.

The Fundamental Problem

The Fundamental Problem is governed by the strain–displacement relationship [5.62], the incremental balance equations [5.70a] and the elastic law [5.78], equipped with the geometric [5.61] and the mechanical [5.70b] boundary conditions.

Once the problem is formulated in terms of displacements, the equations of motion are derived. In the external basis, they become:

$$\left[EAe\bar{x}_j' + \left(\mathring{T} + EAe\right)u_j'\right]' + \tilde{p}_j = m\ddot{u}_j, \qquad j = 1, 2, 3 \qquad [5.79]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and:

$$\mp\left[EAe\bar{x}_j' + \left(\mathring{T} + EAe\right)u_j'\right]_H = \tilde{P}_{jH}, \qquad j = 1, 2, 3 \qquad [5.80]$$

where e is defined in equation [5.64].

When the problem is expressed in the intrinsic reference basis, it appears as follows:

$$\begin{aligned}
&(EAe)' + \left[\left(\mathring{T} + EAe\right)\left(u_t' - \bar{k}u_n\right)\right] \\
&- \bar{k}\left(\mathring{T} + EAe\right)\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right) + \tilde{p}_t = m\ddot{u}_t \\
&(EAe)\,\bar{k} + \left[\left(\mathring{T} + EAe\right)\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right]' \\
&+ \left(\mathring{T} + EAe\right)\left[\bar{k}\left(u_t' - \bar{k}u_n\right) - \bar{\tau}\left(u_b' + \bar{\tau}u_n\right)\right] + \tilde{p}_n = m\ddot{u}_n \\
&\left[\left(\mathring{T} + EAe\right)\left(u_b' + \bar{\tau}u_n\right)\right]' \\
&+ \bar{\tau}\left(\mathring{T} + EAe\right)\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right) + \tilde{p}_b = m\ddot{u}_b
\end{aligned} \qquad [5.81]$$

and:

$$\begin{aligned}
&\mp\left[EAe + \left(\mathring{T} + EAe\right)\left(u_t' - \bar{k}u_n\right)\right]_H = \tilde{P}_t \\
&\mp\left[\left(\mathring{T} + EAe\right)\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right]_H = \tilde{P}_n \\
&\mp\left[\left(\mathring{T} + EAe\right)\left(u_b' + \bar{\tau}u_n\right)\right]_H = \tilde{P}_b
\end{aligned} \qquad [5.82]$$

where e is defined in equation [5.67].

5.3.2 The linearized theory

Linearized theory is based on equations that are *linear* in the displacements measured from a (generally) *nonlinear* equilibrium configuration. Accordingly, the strain–displacement relationship must be linearized in the displacement $\mathbf{u}$, and the incremental balance equations linearized in the increment of stress $\tilde{T} := T - \mathring{T}$, by considering the prestress as an order-1 magnitude.

The strain [5.62], when linearized, becomes:

$$e = \mathbf{u}' \cdot \bar{\mathbf{a}}_t \qquad [5.83]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and the incremental balance equations [5.73], when linearized, become:

$$\begin{aligned} &\left[\tilde{T}\bar{\mathbf{a}}_t + \mathring{T}\mathbf{u}'\right]' + \tilde{\mathbf{p}} = m\ddot{\mathbf{u}} \\ &\mp \left[\tilde{T}\bar{\mathbf{a}}_t + \mathring{T}\mathbf{u}'\right]_H = \tilde{\mathbf{P}}_H \end{aligned} \qquad [5.84]$$

The constitutive law [5.78] is rewritten in such a way to link the incremental stress to the (incremental) strain:

$$\tilde{T} = EAe \qquad [5.85]$$

When these equations are combined, we obtain the linearized equations of motion and mechanical boundary conditions:

$$\begin{aligned} &\left[EA\left(\mathbf{u}' \cdot \bar{\mathbf{a}}_t\right)\bar{\mathbf{a}}_t + \mathring{T}\mathbf{u}'\right]' + \tilde{\mathbf{p}} = m\ddot{\mathbf{u}} \\ &\mp \left[EA\left(\mathbf{u}' \cdot \bar{\mathbf{a}}_t\right)\bar{\mathbf{a}}_t + \mathring{T}\mathbf{u}'\right]_H = \tilde{\mathbf{P}}_H \end{aligned} \qquad [5.86]$$

to be supplemented with the geometric boundary conditions.

In the external basis, the field equations become:

$$\left[EA\left(\sum_{i=1}^{3}\bar{x}_i' u_i'\right)\bar{x}_j' + \mathring{T}u_j'\right]' + \tilde{p}_j = m\ddot{u}_j, \qquad j = 1,2,3 \qquad [5.87]$$

and the boundary conditions become:

$$\mp\left[EA\left(\sum_{i=1}^{3}\bar{x}_i' u_i'\right)\bar{x}_j' + \mathring{T}u_j'\right]_H = \tilde{P}_{jH}, \qquad j = 1,2,3 \qquad [5.88]$$

In the intrinsic basis, we have:

$$\begin{aligned} &\left[EA\left(u_t' - \bar{k}u_n\right)\right]' + \left[\mathring{T}\left(u_t' - \bar{k}u_n\right)\right]' \\ &- \bar{k}\mathring{T}\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right) + \tilde{p}_t = m\ddot{u}_t \\ &EA\bar{k}\left(u_t' - \bar{k}u_n\right) + \left[\mathring{T}\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right]' \\ &+ \mathring{T}\left[\bar{k}\left(u_t' - \bar{k}u_n\right) - \bar{\tau}\left(u_b' + \bar{\tau}u_n\right)\right] + \tilde{p}_n = m\ddot{u}_n \\ &\left[\mathring{T}\left(u_b' + \bar{\tau}u_n\right)\right]' \\ &+ \bar{\tau}\mathring{T}\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right) + \tilde{p}_b = m\ddot{u}_b \end{aligned} \qquad [5.89]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with:

$$\begin{aligned} &\mp \left[EA\left(u_t' - \bar{k}u_n\right) + \mathring{T}\left(u_t' - \bar{k}u_n\right)\right]_H = \tilde{P}_t \\ &\mp \left[\mathring{T}\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right]_H = \tilde{P}_n \\ &\mp \left[\mathring{T}\left(u_b' + \bar{\tau}u_n\right)\right]_H = \tilde{P}_b \end{aligned} \tag{5.90}$$

These equations govern the small amplitude motions of the cable around the prestressed configuration[9].

REMARK 5.14. Equations [5.84] suggest the following consideration. According to the linearized theory (and consistently with the Leibniz rule for the derivative of a product), the internal force equilibrating the incremental loads (and inertia forces) is the sum of two contributions: *the change of stress acting in the old geometry*, i.e. $EA\left(\mathbf{u}' \cdot \bar{\mathbf{a}}_t\right)\bar{\mathbf{a}}_t$, and *the old stress acting in the change of geometry*, i.e. $\mathring{T}\mathbf{u}'$. The first contribution determines the *elastic stiffness* and the second contribution determines the *geometric stiffness*.

5.3.3 Taut strings

An idealized model of prestressed cable is the *taut string*. It is assumed that the cable, in the reference configuration, is solicited exclusively by equal and opposite end-forces, namely $-\mathring{\mathbf{P}}_A = \mathring{\mathbf{P}}_B =: \mathring{T}\mathbf{i}_1$ (i.e. the self-weight is not present, Figure 5.3). These forces can be active or reactive. In the first case, the string is free in the space with two opposite forces attached at the ends; in the second case, the string is first stretched between two fixed supports, whose distance is larger than the natural length, and then the ends are fixed. A mixed free-fixed set-up is also possible. Of course, free-strings can only support self-equilibrated incremental loads (even of inertial nature), while constrained strings can support more general force systems. Since $\mathring{\mathbf{p}} = \mathbf{0}$ in equations [5.59], $\mathring{\mathbf{t}}$ is constant along s, and therefore, since it is everywhere tangent to the centerline, *the cable is rectilinear*. From the boundary conditions, it follows that $\mathring{\mathbf{t}} = \mathring{T}\mathbf{i}_1$, i.e. $\mathring{T}$ is the tension of the taut string. In the reference configuration, the intrinsic triad $\bar{\mathcal{B}}_f$ (there exist infinite amounts of them) can be conveniently taken to be coincident with the external basis $\mathcal{B}_e$. Moreover, due to the straightness, the Frenet curvature and the torsion identically vanish, i.e. $\bar{k} = \bar{\tau} = 0$.

When incremental loads (possibly including the self-weight) act on the string, this moves into a new (current) configuration, which is generally curved and described by

9. With the symbols of the metamodel, the linearized equations of motion become $\boldsymbol{Lu} + \boldsymbol{Gu} = \tilde{\boldsymbol{p}}$, in the field, and $\boldsymbol{\mathcal{L}}_H\boldsymbol{u} + \boldsymbol{\mathcal{G}}_H\boldsymbol{u} = \tilde{\boldsymbol{P}}$, on the boundary.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$\mathbf{x}(s,t)$, defining a current Frenet triad $\mathcal{B}_f$. All the equations governing the motion of the string can be obtained by specializing those obtained for the cable, as illustrated later.

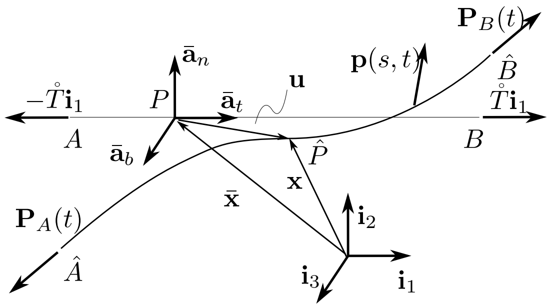


Figure 5.3: Taut string: pretension $\mathring{T}$, preloads $\pm\mathring{T}\mathbf{i}_1$, loads $\mathbf{p}$, $\mathbf{P}_H$, displacement $\mathbf{u}$.

Nonlinear model

The strain–displacement relationship [5.62] becomes:

$$e = \sqrt{1 + 2\mathbf{u}' \cdot \mathbf{i}_1 + \mathbf{u}' \cdot \mathbf{u}'} - 1 \qquad [5.91]$$

or, since $\mathbf{u} := \sum_{j=1}^{3} u_j \mathbf{i}_j$:

$$e = \sqrt{1 + 2\left(u_1' + \frac{1}{2}\sum_{j=1}^{3} u_j'^2\right)} - 1 \qquad [5.92]$$

The (quasi-exact) incremental balance equations [5.73], with $\bar{\mathbf{a}}_t \equiv \mathbf{i}_1$, become:

$$\begin{aligned} &\left[\tilde{T}\mathbf{i}_1 + T\mathbf{u}'\right]' + \tilde{\mathbf{p}} = m\ddot{\mathbf{u}} \\ &\mp \left[\tilde{T}\mathbf{i}_1 + T\mathbf{u}'\right]_H = \tilde{\mathbf{P}}_H \end{aligned} \qquad [5.93]$$

The elastic law [5.78] remains unaltered:

$$T = \mathring{T} + EAe \qquad [5.94]$$

When all these equations are combined among them, the following equations of motion are derived:

$$\begin{aligned} &\mathring{T}\mathbf{u}'' + \left[EAe\left(\mathbf{i}_1 + \mathbf{u}'\right)\right]' + \tilde{\mathbf{p}} = m\ddot{\mathbf{u}} \\ &\mp \left[\mathring{T}\mathbf{u}' + EAe\left(\mathbf{i}_1 + \mathbf{u}'\right)\right]_H = \tilde{\mathbf{P}}_H \end{aligned} \qquad [5.95]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with e given by equation [5.92].

The projection of the balance equations onto the external basis leads to:

$$\begin{aligned} &\mathring{T}u_1'' + [EAe\,(1+u_1')]' + \tilde{p}_1 = m\ddot{u}_1 \\ &\mathring{T}u_2'' + (EAeu_2')' + \tilde{p}_2 = m\ddot{u}_2 \\ &\mathring{T}u_3'' + (EAeu_3')' + \tilde{p}_3 = m\ddot{u}_3 \end{aligned} \qquad [5.96]$$

together with:

$$\begin{aligned} &\mp\left[\mathring{T}u_1' + EAe\,(1+u_1')\right]_H = \tilde{P}_{1H} \\ &\mp\left[\left(\mathring{T} + EAe\right)u_2'\right]_H = \tilde{P}_{2H} \\ &\mp\left[\left(\mathring{T} + EAe\right)u_3'\right]_H = \tilde{P}_{3H} \end{aligned} \qquad [5.97]$$

where e is defined by equation [5.92].

REMARK 5.15. Although the equation of motion of a taut string are much simpler than the equation for a cable, they are still *fully coupled*, since the strain depends on all the components of motion.

Linearized theory

In the framework of the linearized theory, the strain [5.92] must be linearized as:

$$e = u_1' \qquad [5.98]$$

as well as the incremental balance equations [5.93], which become:

$$\begin{aligned} &\tilde{T}'\mathbf{i}_1 + \mathring{T}\mathbf{u}'' + \tilde{\mathbf{p}} = m\ddot{\mathbf{u}} \\ &\mp\left[\tilde{T}\mathbf{i}_1 + \mathring{T}\mathbf{u}'\right]_H = \tilde{\mathbf{P}}_H \end{aligned} \qquad [5.99]$$

while the constitutive law is presented in the incremental form:

$$\tilde{T} = EAe \qquad [5.100]$$

By combining the previous equations, the linearized equations of motion are obtained:

$$\begin{aligned} &EAu_1''\mathbf{i}_1 + \mathring{T}\mathbf{u}'' + \tilde{\mathbf{p}} = m\ddot{\mathbf{u}} \\ &\mp\left[EAu_1'\mathbf{i}_1 + \mathring{T}\mathbf{u}'\right]_H = \tilde{\mathbf{P}}_H \end{aligned} \qquad [5.101]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

When these are projected on the external basis, they provide:

$$\left(EA+\overset{\circ}{T}\right)u_1''+\tilde{p}_1=m\ddot{u}_1$$
$$\overset{\circ}{T}u_2''+\tilde{p}_2=m\ddot{u}_2 \qquad [5.102]$$
$$\overset{\circ}{T}u_3''+\tilde{p}_3=m\ddot{u}_3$$

with the mechanical boundary conditions:

$$\mp\left[\left(EA+\overset{\circ}{T}\right)u_1'\right]_H=\tilde{P}_{1H}$$
$$\mp\left[\overset{\circ}{T}u_2'\right]_H=\tilde{P}_{2H} \qquad [5.103]$$
$$\mp\left[\overset{\circ}{T}u_3'\right]_H=\tilde{P}_{3H}$$

alternative to geometric boundary conditions.

REMARK 5.16. The linearized equations of motion of the taut string are *all uncoupled*. Thus, longitudinal and transverse motions are each independent of other.

REMARK 5.17. Each of equations [5.102] has the form of the *linear wave equation* $\ddot{u}=c_{l,t}^2u''$, where $c_l:=\sqrt{\left(EA+\overset{\circ}{T}\right)/m}$ and $c_t:=\sqrt{\overset{\circ}{T}/m}$ are the *longitudinal and transverse celerity* of the traveling waves, respectively.

REMARK 5.18. In the linearized theory, the elastic stiffness of the taut string, $EA\partial_s^2$, only contributes to the longitudinal equation of motion, while the transverse equations involve the geometrical stiffness $\overset{\circ}{T}\partial_s^2$. On the other hand, since, usually, $\overset{\circ}{T}/EA\ll 1$, the longitudinal motion of the taut string is essentially unaffected by the prestress; consistently, $c_l=\sqrt{\frac{EA}{m}}$.

5.4 Shallow cables

So far we developed exact or quasi-exact models of flexible cables, leading to quite complex equations. Now, we want to show how to derive an *approximated* model of *shallow cable*, prestressed by its own weight and, possibly, by end forces. Derivation is mainly based on the results of [IRV 74, IRV 81, PER 87]. Shallowness is meant here as a small deviation of the cable from the chord that joins the two (assumed fixed) ends; the chord can be horizontal (suspended cable) or inclined (cable-stay). We will refer to the maximum normal deviation as the *sag* of the cable. Moreover, we will introduce hypotheses on the order of magnitude of the displacements, and we will expand the strain in series, in order to obtain a more amenable expression.

We mainly carry out the analysis in the intrinsic reference basis; however, at the end, we will outline how to perform similar calculations in the external basis.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

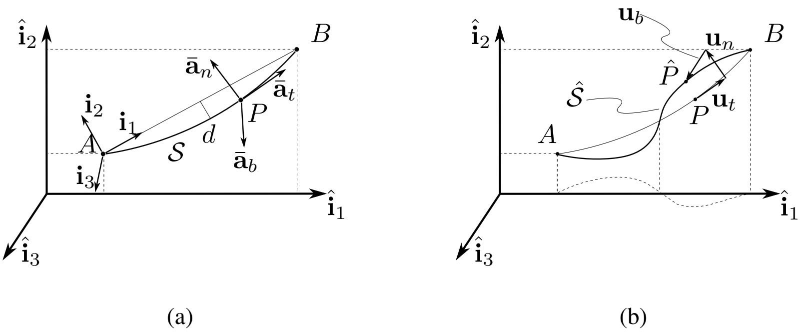


Figure 5.4: Shallow cable: (a) reference configuration; centerline $\mathcal{S}$ in the $(\mathbf{i}_1, \mathbf{i}_2)$-plane, sag d; (b) current configuration, centerline $\hat{S}$ and its horizontal projection, displacement components u_t, u_n, u_b.

5.4.1 An approximated nonlinear model

Hypotheses

We introduce the following hypotheses:

1) In the reference configuration, *the cable is planar*, and possesses a small sag-to-length ratio $\delta := d/l$, e.g. of the order $\mathrm{O}\left(10^{-1}\right)$; planarity entails that the torsion $\bar{\tau}$ identically goes to zero.

2) *The Frenet curvature $\bar{k}$ is assumed constant* along the cable and small, namely $\bar{k}l = \mathrm{O}\left(\delta^2\right)$, as for a shallow parabola.

3) *The prestress $\mathring{T}$ is assumed constant* along the cable[10].

4) The $\mathring{T}/EA$ ratio is small, e.g. of order $\mathrm{O}\left(10^{-3}\right)$ or smaller; this entails that the transverse celerity of the (rectified) cable, $c_t = \sqrt{\mathring{T}/m}$, is much more smaller than the longitudinal celerity $c_l = \sqrt{EA/m}$.

5) All the displacements vary on a scale-length of the order of the cable length, i.e. $u'_\alpha = \mathrm{O}\left(u_a/l\right)$.

6) The transverse displacements u_n, u_b are both assumed of the order of the sag, so that $u_n/l = \mathrm{O}\left(\delta\right)$, $u_b/l = \mathrm{O}\left(\delta\right)$; the tangential displacement u_t, instead, is assumed to be smaller, i.e. $u_t/l = \mathrm{O}\left(\delta^2\right)$.

7) The incremental loads are purely transverse, i.e. $\tilde{p}_t = 0$.

10. In a horizontal cable under self weight p, $\mathring{T}$ is close to the value $pl^2/(8d)$; in an inclined cable, $\mathring{T} \simeq p_2 l^2/(8d) + p_1 l f(s)$, where p_1 and p_2 are the components of p in the triad intrinsic to the chord (see figure 5.4(a)), and $f(s) = \mathrm{O}(1)$, so that, due to the tangential component, the tension is variable. However, very often inclined cables are prestressed by end forces (as it happens in stays), so that a constant contribution, usually larger, adds itself to the former one, making the approximation acceptable. Moreover, stays are much more shallow, e.g. they have $\delta = \mathrm{O}(10^{-2})$, so that also the constant contribution to $\mathring{T}$ due to p_2 prevails on that due to p_1.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The expanded strain–displacement relationship

By expanding the unit strain [5.67] in series for small displacements, we have:

$$e = u_t' - \bar{k} u_n + \frac{1}{2}\left[\left(u_t' - \bar{k} u_n\right)^2 + \left(u_n' + \bar{k} u_t\right)^2 + u_b'^2\right] + \text{h.o.t.} \qquad [5.104]$$

where we accounted for $\bar{\tau} = 0$. Since the first term in the square bracket is the squared linear term, we ignore it (Biot approximation); moreover, we neglect $\bar{k} u_t$ with respect u_n', since their ratio is $\mathrm{O}\left(\delta^2\right)$; by ignoring higher order terms, we finally write:

$$e = u_t' - \bar{k} u_n + \frac{1}{2}\left(u_n'^2 + u_b'^2\right) \qquad [5.105]$$

The static condensation of the tangential displacement

Firstly, we analyze the balance equations [5.81a], which govern the tangential motion. We observe that the leading term among the internal forces is $(EAe)' = \mathrm{O}\left(EAu_t''\right)$, the remaining terms being δ^2-times smaller; moreover, if we are interested in time-evolutions having a characteristic time $\mathrm{O}\left(l/c_t\right)$ (as it occurs when the motion is prevalently transverse), we can conclude that:

$$m\ddot{u}_t = \mathrm{O}\left(m u_t \frac{c_t^2}{l^2}\right) = \mathrm{O}\left(\frac{\mathring{T} u_t}{l^2}\right) \ll \mathrm{O}\left(\frac{EAu_t}{l^2}\right) = (EAe)' \qquad [5.106]$$

Therefore, the tangential inertia can be neglected, and the equation simplified into:

$$(EAe)' = 0 \qquad [5.107]$$

From this, it follows that:

$$e\left(s,t\right) = e_0\left(t\right) \qquad [5.108]$$

i.e. *the unit strain is constant along the cable*, and it varies only with time. Of course, this result, which is consequent to strong simplifications, should be understood in the sense that the strain is *weakly variable with* s, so that, in a rough approximation, it is taken as a constant.

By integrating the strain–displacement relationship [5.105], we get:

$$u_t = u_{tA}\left(t\right) + e_0\left(t\right)\left(s - s_A\right) + \bar{k}\int_{s_A}^{s} u_n ds - \frac{1}{2}\int_{s_A}^{s}\left(u_n'^2 + u_b'^2\right) ds \qquad [5.109]$$

This expresses the tangential displacement as a “slave” of the transverse displacements u_n, u_b. We therefore say that the tangential displacement has been *statically condensed*.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Equation [5.109] depends on two arbitrary functions of time (i.e constant in space), namely u_{tA} and e_0, that must be determined by geometric boundary conditions. The simplest case is that of fixed ends, namely $u_{tA} = u_{tB} = 0,\ \forall t$, but other conditions prescribing the motion of the supports can easily be accounted for (e.g. [LUO 12]). By enforcing homogeneous conditions, the strain is found:

$$e_0(t) = -\frac{\bar{k}}{l}\int_{s_A}^{s_B} u_n ds + \frac{1}{2l}\int_{s_A}^{s_B}\left(u_n'^2 + u_b'^2\right) ds \qquad [5.110]$$

The transverse motion

We now move on to analyze the balance equations along the normal and binormal, equations [5.81b,c], in which $\bar{\tau} = 0$ must be taken, together with $\mathring{T} + EAe = \text{const}$, as follows from hypothesis 3 and the previous analysis. No further simplifications can be done in the equation relevant to the binormal directions. Instead, concerning the equation in the normal direction, we observe that $\bar{k}u_t/u_n' = \mathrm{O}\left(\delta^2\right)$, as well as $\bar{k}\left(u_t' - \bar{k}u_n\right)/u_n'' = \mathrm{O}\left(\delta^2\right)$, so that the smaller terms can be neglected. Therefore, to within an error of $\mathrm{O}\left(\delta^2\right)$, the transverse motion is governed by the following, simplified, equations:

$$\begin{aligned} &\left(\mathring{T} + EAe_0\right)u_n'' + EAe_0\bar{k} + \tilde{p}_n = m\ddot{u}_n \\ &\left(\mathring{T} + EAe_0\right)u_b'' + \tilde{p}_b = m\ddot{u}_b \end{aligned} \qquad [5.111]$$

where, by virtue of equations [5.108], $e = e_0(t)$ is given by equation [5.110]. Hence, the motion is governed by two integro-differential equations in the transverse displacements $u_n(s,t),\ u_b(s,t)$. The problem is completed by the geometric boundary conditions that have not yet been used; for fixed ends, they become:

$$u_{nH} = u_{bH} = 0,\ \forall t \qquad [5.112]$$

The formulation in the external basis

If components are desired in the external basis, instead of the intrinsic basis, hypothesis 2, concerning the initial curvature, must be substituted by the (equivalent) hypothesis that the arc element ds can be confused with its along-chord projection dx_1; moreover, the same hypotheses introduced for the tangential u_t and transverse u_n, u_b displacements must be made for the *along-chord* u_1 and *transverse-to-chord* displacements u_2, u_3, respectively (with minor physical meaning).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The strain [5.64] admits the MacLaurin expansion:

$$e = \sum_{j=1}^{3} \bar{x}_j' u_j' + \frac{1}{2} \sum_{j=1}^{3} u_j'^2 + \text{h.o.t.} \tag{5.113}$$

Since the cable is planar in the reference configuration, $\bar{x}_3 \equiv 0$. Moreover, if $x_1' \simeq 1$ is taken, and $u_1'^2$ is neglected with respect to u_1', the strain becomes:

$$e = u_1' + \bar{x}_2' u_2' + \frac{1}{2} \left(u_2'^2 + u_3'^2 \right) \tag{5.114}$$

With the same hypotheses, the along-chord equation of motion [5.79a] simplifies into:

$$\left[EAe \left(1 + u_1' \right) + \mathring{T} u_1' \right]' = m\ddot{u}_1 \tag{5.115}$$

Since $u_1' = \mathrm{O}\left(e\right)$, we can neglect it in the first term; then, since $\mathring{T} \ll EA$, we can neglect $\mathring{T} u_1'$ with respect to EAe; finally, by using arguments already discussed, we can ignore the along-chord inertia forces, so that we find again equation [5.107], from which $e\left(s,t\right) = e_0\left(t\right)$. By using the expression [5.114] for e and integrating it under homogeneous boundary conditions, we find:

$$e_0\left(t\right) = \frac{1}{l} \int_{s_A}^{s_B} \bar{x}_2' u_2' ds + \frac{1}{2l} \int_{s_A}^{s_B} \left(u_2'^2 + u_3'^2 \right) ds \tag{5.116}$$

The two remaining equations of motion, consequently, become:

$$\begin{aligned} &\mathring{T} u_2'' + EAe_0 \left(\bar{x}_2'' + u_2'' \right) + \tilde{p}_2 = m\ddot{u}_2 \\ &\left(\mathring{T} + EAe_0 \right) u_3'' + \tilde{p}_3 = m\ddot{u}_3 \end{aligned} \tag{5.117}$$

which, together with the boundary conditions:

$$u_{2H} = u_{3H} = 0, \ \forall t \tag{5.118}$$

govern the motion.

5.4.2 An approximated linearized model

If we are interested in analyzing motions of small amplitude around the prestressed configuration of a shallow cable, we can resort to the linearized theory. Accordingly, only the linear part of the unit strain, equations [5.105] or [5.114], is taken, and second-order products between increments of stress and geometry changes are ignored in the incremental balance equations.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

As a result, the transverse equations of motion, in intrinsic components [5.111], simplify into:

$$\begin{aligned} &\mathring{T}u_n'' + EAe_0\bar{k} + \tilde{p}_n = m\ddot{u}_n \\ &\mathring{T}u_b'' + \tilde{p}_b = m\ddot{u}_b \end{aligned} \qquad [5.119]$$

in which, after the linearization of equation [5.110]:

$$e_0\left(t\right) = -\frac{\bar{k}}{l}\int_{s_A}^{s_B} u_n ds \qquad [5.120]$$

Similarly, the transverse equations of motion [5.117], in Cartesian components, become:

$$\begin{aligned} &\mathring{T}u_2'' + EAe_0\bar{x}_2'' + \tilde{p}_2 = m\ddot{u}_2 \\ &\mathring{T}u_3'' + \tilde{p}_3 = m\ddot{u}_3 \end{aligned} \qquad [5.121]$$

where, by linearizing equation [5.116]:

$$e_0\left(t\right) = \frac{1}{l}\int_{s_A}^{s_B} \bar{x}_2' u_2' ds \qquad [5.122]$$

REMARK 5.19. In the framework of the linearized theory, the out-of-plane motion of shallow cable is governed by the equation of the taut string [5.102], i.e. any information concerning the curvature of the cable is lost, in the simplification process. In contrast, the in-plane motion is governed by a modified equation, which accounts for the pre-existing curvature, $\bar{k}$ or $\bar{x}_2''$. No such term can appear in the out-of-plane equation of motion, since the pre-existing torsion $\bar{\tau}$ is zero.

5.5 Inextensible cables

So far, we considered flexible cables endowed with a finite axial stiffness, so that the potential elastic energy accounts solely for unit extension. However, a simpler model can be formulated, in which *even the extensional energy is considered small in comparison with the potential energy of the external loads*, i.e. with the work spent by the forces in changing the cable configuration. According to this model, the cable behaves as a perfectly flexible body against bending and torsion, but infinitely stiff against extension. We will call it the *inextensible cable*. As we did for internally constrained beams (Chapter 4), we derive this model from the unconstrained beam.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

REMARK 5.20. While the inextensible model is expected to capture the essential aspect of *slack cables*, in contrast it fails to accurately describe the mechanics of shallow cables, where the elastic energy is not smaller than the work spent by the external forces. As a matter of fact, if we put $e_0 = 0$ in equations [5.111] or [5.117], we see that any information related to the curvature of the cable disappears, so that these equations reduce to those of the taut string. Moreover, it has been shown in [IRV 74, IRV 81], that the inextensible model fails to describe the free evolution of elastic shallow cables even in the linear range.

5.5.1 Inextensible unprestressed cables

The inextensibility condition requires that $\mathbf{e} = \mathbf{0}$ at any point and at any time; consequently, the unit strain identically goes to zero, i.e. $e := \|\mathbf{x}'\| - 1 = 0$. This constitutes a geometric constraint limiting the configurations admissible by the cable; by using Cartesian components, the constraint becomes:

$$e := \sqrt{x_1'^2 + x_2'^2 + x_3'^2} - 1 = 0 \qquad [5.123]$$

or, in a more expressive form:

$$x_1'^2 + x_2'^2 + x_3'^2 = 1 \qquad [5.124]$$

Since the stretch $\lambda := d\hat{s}/ds \equiv \|\mathbf{x}'\| = 1$, the unit vectors of the Frenet triad in the current configuration become as in the natural parametrization (compare them with equations [5.9]), i.e.:

$$\mathbf{a}_t = \mathbf{x}', \qquad \mathbf{a}_n = \frac{1}{k}\mathbf{x}'', \qquad \mathbf{a}_b = \frac{1}{k}\left(\mathbf{x}' \times \mathbf{x}''\right) \qquad [5.125]$$

On the other hand, the VPP [5.27], with equation [5.24], states that:

$$\int_{\mathcal{S}} \mathbf{p} \cdot \mathbf{v} ds + \sum_{H=A}^{B} \mathbf{P}_H \cdot \mathbf{v}_H = \int_{\mathcal{S}} \mathbf{t} \cdot \mathbf{R}\dot{\mathbf{e}} ds \qquad [5.126]$$

for any admissible virtual motions. Since the latter must satisfy the constraint $\dot{\mathbf{e}} = \mathbf{0}$, the internal virtual power identically goes to zero. However, according to the mixed formulation, we can consider the *reactive stress* $\mathbf{t}$ as a Lagrangian (vector) parameter spending virtual power in the zero strain-rate, so that the relevant term works as an appended constraint, and the previous principle still holds. As a result, the same balance equations [5.31] and boundary conditions [5.30] valid for the extensible cable are obtained:

$$\begin{aligned} &\mathbf{t}' + \mathbf{p} = m\ddot{\mathbf{x}} \\ &(\mathbf{P}_H \pm \mathbf{t}_H) \cdot \mathbf{v}_H = 0 \end{aligned} \qquad [5.127]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

in which, however, now $\mathbf{t} := T\mathbf{a}_t = T\mathbf{x}'$. When these equations are projected in the external basis, they provide (see equations [5.36] and [5.37]):

$$\begin{aligned} &\frac{\partial}{\partial s}\left(T\frac{\partial x_1}{\partial s}\right) + p_1 = m\ddot{x}_1 \\ &\frac{\partial}{\partial s}\left(T\frac{\partial x_2}{\partial s}\right) + p_2 = m\ddot{x}_2 \\ &\frac{\partial}{\partial s}\left(T\frac{\partial x_3}{\partial s}\right) + p_3 = m\ddot{x}_3 \end{aligned} \qquad [5.128]$$

and the mechanical:

$$\mp\left[T\frac{\partial x_j}{\partial s}\right]_H = P_{jH}, \qquad j = 1, 2, 3 \qquad [5.129]$$

or geometric boundary conditions:

$$x_{jH} = \breve{x}_{jH}, \qquad j = 1, 2, 3 \qquad [5.130]$$

The Fundamental Problem is completed by the constraint [5.124]. Thus, we have four scalar equations in four unknowns, namely the three components of the position vector x_j and the reactive tension T.

5.5.2 Inextensible prestressed cables

The nonlinear theory

Let us consider the inextensible cable in the reference configuration, described by $\bar{\mathbf{x}}(s)$. When the cable moves to a current configuration $\mathbf{x} = \mathbf{x}(s, t)$, it must satisfy the constraint $e = \|\mathbf{x}'\| - 1 = 0$. If we introduce displacements $\mathbf{u} := \mathbf{x} - \bar{\mathbf{x}}$, and account for $\bar{\mathbf{x}}' = \bar{\mathbf{a}}_t$, the constraint becomes:

$$\|\bar{\mathbf{a}}_t + \mathbf{u}'\| = 1 \qquad [5.131]$$

or, in Cartesian components (see equation [5.64]):

$$\sum_{j=1}^{3} \bar{x}_j' u_j' + \frac{1}{2}\sum_{j=1}^{3} u_j'^2 = 0 \qquad [5.132]$$

or, in the reference intrinsic components (see equation [5.67]):

$$u_t' - \bar{k}u_n + \frac{1}{2}\left[\left(u_t' - \bar{k}u_n\right)^2 + \left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)^2 + \left(u_b' + \bar{\tau}u_n\right)^2\right] = 0 \quad [5.133]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Let us assume that, in the reference configuration, the cable is in equilibrium under *static preloads* $\mathring{\mathbf{p}}, \mathring{\mathbf{P}}_H$ causing the prestress $\mathring{\mathbf{t}}$ and that, in the current configuration, *dynamic total forces* $\mathbf{p}$, $\mathbf{P}_H$ trigger the stress $\mathbf{t}$. We can define *incremental loads* $\tilde{\mathbf{p}} := \mathbf{p} - \mathring{\mathbf{p}}$ and $\tilde{\mathbf{P}}_H := \mathbf{P}_H - \mathring{\mathbf{P}}_H$, and *incremental stress* $\tilde{\mathbf{t}} := \mathbf{t} - \mathring{\mathbf{t}}$ to reformulate the balance equations in incremental form. As we saw in section 5.3 for the extensible cable, they become (equations [5.70]):

$$\begin{aligned} &\left(\mathbf{t} - \mathring{\mathbf{t}}\right)' + \tilde{\mathbf{p}} = m\ddot{\mathbf{u}} \\ &\mp \left[\left(\mathbf{t} - \mathring{\mathbf{t}}\right)\right]_H = \tilde{\mathbf{P}}_H \end{aligned} \qquad [5.134]$$

Now, however, differently from that case, the incremental vector stress becomes (compare it with equation [5.72]):

$$\mathbf{t} - \mathring{\mathbf{t}} = T\left(\bar{\mathbf{a}}_t + \mathbf{u}'\right) - \mathring{T}\bar{\mathbf{a}}_t \qquad [5.135]$$

so that *the approximation* $\lambda = 1$ *has not to be used*, since it is intrinsic to the model of the inextensible cable. Therefore, the Cartesian forms [5.74] and [5.75] and the intrinsic forms [5.76] and [5.77] of the incremental balance equations [5.134] follow.

In summary, the Fundamental Problem, is governed by three balance equations and a geometrical constraint; the unknowns are the three displacement components and the reactive stress. In Cartesian components, by letting $T = T_0 + \tilde{T}$, the problem becomes:

$$\begin{aligned} &\left[\tilde{T}\bar{x}_j' + Tu_j'\right]' + \tilde{p}_j = m\ddot{u}_j, \qquad j = 1, 2, 3 \\ &\sum_{j=1}^{3} \bar{x}_j' u_j' + \frac{1}{2}\sum_{j=1}^{3} u_j'^2 = 0 \end{aligned} \qquad [5.136]$$

and:

$$\begin{aligned} &\mp \left[\tilde{T}\bar{x}_j' + Tu_j'\right]_H = \tilde{P}_{jH}, \qquad j = 1, 2, 3 \\ &u_{jH} = \breve{u}_{jH}, \qquad j = 1, 2, 3 \end{aligned} \qquad [5.137]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

In intrinsic components, it is expressed by:

$$\begin{aligned}
&\tilde{T}' + T'\left(u_t' - \bar{k}u_n\right) \\
&+ T\left[\left(u_t' - \bar{k}u_n\right)' - \bar{k}\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right] + \tilde{p}_t = m\ddot{u}_t \\
&\tilde{T}\bar{k} + T'\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right) \\
&+ T\left[\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)' + \bar{k}\left(u_t' - \bar{k}u_n\right) - \bar{\tau}\left(u_b' + \bar{\tau}u_n\right)\right] + \tilde{p}_n = m\ddot{u}_n \\
&T'\left(u_b' + \bar{\tau}u_n\right) + T\left[\left(u_b' + \bar{\tau}u_n\right)' + \bar{\tau}\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right] + \tilde{p}_b = m\ddot{u}_b \\
&u_t' - \bar{k}u_n + \frac{1}{2}\left[\left(u_t' - \bar{k}u_n\right)^2 + \left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)^2 + \left(u_b' + \bar{\tau}u_n\right)^2\right] = 0
\end{aligned} \qquad [5.138]$$

with:

$$\begin{aligned}
&\mp\left[\tilde{T} + T\left(u_t' - \bar{k}u_n\right)\right]_H = \tilde{P}_t \\
&\mp\left[T\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right]_H = \tilde{P}_n \\
&\mp\left[T\left(u_b' + \bar{\tau}u_n\right)\right]_H = \tilde{P}_b \\
&u_t = \breve{u}_t, \qquad u_n = \breve{u}_n, \qquad u_b = \breve{u}_b
\end{aligned} \qquad [5.139]$$

The linearized theory

When the constraint [5.62] and the incremental balance equations [5.134] are linearized, they provide the governing equation for the linearized theory, ruling the small motions of the cable around the prestressed configuration. They become:

$$\begin{aligned}
&\left[\tilde{T}\bar{\mathbf{a}}_t + \mathring{T}\mathbf{u}'\right]' + \tilde{\mathbf{p}} = m\ddot{\mathbf{u}} \\
&\mp\left[\tilde{T}\bar{\mathbf{a}}_t + \mathring{T}\mathbf{u}'\right]_H = \tilde{\mathbf{P}}_H \\
&\mathbf{u}' \cdot \bar{\mathbf{a}}_t = 0
\end{aligned} \qquad [5.140]$$

In Cartesian components, they become:

$$\begin{aligned}
&\left(\tilde{T}\bar{x}_j' + \mathring{T}u_j'\right)' + \tilde{p}_j = m\ddot{u}_j \\
&\mp\left[\tilde{T}\bar{x}_j' + \mathring{T}u_j'\right]_H = \tilde{P}_{jH}, \qquad j = 1, 2, 3 \\
&\sum_{j=1}^{3} \bar{x}_j' u_j' = 0
\end{aligned} \qquad [5.141]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

In the intrinsic basis, we have:

$$\begin{aligned}
&\tilde{T}' - \mathring{T}\bar{k}\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right) + \tilde{p}_t = m\ddot{u}_t \\
&\tilde{T}\bar{k} + \mathring{T}'\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right) \\
&\quad + \mathring{T}\left[\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)' - \bar{\tau}\left(u_b' + \bar{\tau}u_n\right)\right] + \tilde{p}_n = m\ddot{u}_n \\
&\mathring{T}'\left(u_b' + \bar{\tau}u_n\right) \\
&\quad + \mathring{T}\left[\left(u_b' + \bar{\tau}u_n\right)' + \bar{\tau}\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right] + \tilde{p}_b = m\ddot{u}_b \\
&u_t' - \bar{k}u_n = 0
\end{aligned} \qquad [5.142]$$

with:

$$\begin{aligned}
&\mp\tilde{T}_H = \tilde{P}_t \\
&\mp\left[\mathring{T}\left(u_n' + \bar{k}u_t - \bar{\tau}u_b\right)\right]_H = \tilde{P}_n \\
&\mp\left[\mathring{T}\left(u_b' + \bar{\tau}u_n\right)\right]_H = \tilde{P}_b
\end{aligned} \qquad [5.143]$$

in which the kinematic constraint has been exploited, in order to vanish some terms in the balance equations.

5.6 Summary

In this chapter, we discussed several models of *flexible cable*. We first performed an order of magnitude analysis, based on energy considerations, to validate the common idea that a cable is an extremely slender beam. The limit process, carried out under the hypothesis that the displacement field does not vary on a too short scale, led us to formulate a model of a 1D body endowed with axial stiffness only, and zero bending and torsion stiffness. Such a body is modeled as a 1D Cauchy continuum, embedded in a 3D-space.

Firstly, *unprestressed cables* were addressed, for which the reference configuration is taken coincident with one of the infinite natural configurations that the cable can assume. However, the arbitrary choice of one of them revealed itself to be inessential, once kinematics are expressed in terms of the (absolute) current position, instead of the (relative) displacement. Since the continuum is non-polar, Frenet triads, macro-rotations and macro-spin had to be introduced, based on the state of a *neighborhood of the point*, instead of the state of the point itself. The analysis led us to define *the strain as a vector which is tangent to the centerline in the reference configuration*. Its modulus is the unit strain, which is the unique component

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

of the strain in the reference triad (i.e. there are not shear-strains). The strain-rate is affected only by the tangential component of the velocity gradient (i.e. its stretching part), since the transverse component is responsible for the macro-spin. Such a result required introducing a dual variable, which is a *stress-force tangent to the centerline in the current configuration*. Its modulus is the tension of the cable, which is the unique component of the stress in the current triad (i.e. there are not shear-stresses). By the VPP, we obtained a vector balance equation that can be recognized as the linear momentum equation of the alternative force balance approach. We commented that the angular momentum equation does not appear in the formulation, since, due to the lack of polar contributions, it is trivially satisfied by the fact that the stress-force is tangent to the centerline. The projection of the balance equations on the principal triad, in the static case, revealed interesting mechanical aspects: namely the cable locally disposes itself in a (osculating) plane that contains the local force; therefore, if the system of forces is planar, the cable is also contained in the same plane. Linear 1D constitutive laws were successively given, for hyperelastic and viscoelastic materials. Finally, the elastic Fundamental Problem was formulated, and the governing equations expressed in the Cartesian basis.

Prestressed cables were successively addressed, i.e. cables in equilibrium under static forces, disturbed by incremental forces. We took the prestressed configuration as reference configuration, and measured all quantities from it. Therefore, we expressed the strain in terms of displacements, and defined incremental loads and stresses. We derived an *incremental balance equation* by subtracting the balance equations relevant to the current and the reference configurations. However, we noted, in order that the scalar increment of tension appears in the equation, a simplification must be introduced, namely to *neglect the stretch* in the current balance equations. This hypothesis seems reasonable in applications, since the stretch is nearly equal to 1; however, strictly speaking, the model is inconsistent, and can be justified only on the grounds of an asymptotic analysis. For this reason, we call the model *quasi-exact*. We formulated the elastic Fundamental Problem both in Cartesian components and in the intrinsic components, in the (known) Frenet basis of the reference configuration. These equations were successively linearized, to obtain a model able to describe motions of small amplitudes around the prestressed configuration.

The general equations for prestressed cables were then specialized to *taut strings*, i.e. to cables that, in their reference configuration, are solicited by two equal and opposite forces, so that they assume a rectilinear configuration. The relevant equations are considerably simpler, yet coupled. However, when they are linearized, a full uncoupling occurs, and they reduce to the well-known traveling wave (linear) equations. It was noted, that the celerity of longitudinal waves is much higher that that of transverse waves.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

An approximated model for *shallow cables* was then derived. These are prestressed cables that assume a curvilinear shape in their reference configuration that is close to the chord that joins the two ends. The commonest case of *planar* reference configuration was addressed. By defining the sag as the maximum normal distance between the chord and the centerline, we can say that the cable is shallow if the sag-to-chord length is of the order of 1/10. For these cables, a technical theory has been developed in the literature, based on quite strong hypotheses that, however, have been validated by numerical results. Due to the shallowness of the cable, the prestress and the curvature of the cable are assumed constant; moreover, the tangential displacement is assumed smaller than the transverse component; finally, the tangential inertia force is neglected, allowing a *static condensation* of the tangential displacement. As a result, the unit strain is found to be constant along the cable. The relevant equations of motion are of integro-differential type, and only contain the transverse components of motion. If these equations are linearized, the out-of-plane motion turns out to be uncoupled from the in-plane motion, and governed by the simpler taut string equation, i.e. any information about the curvature of the cable is lost. The in-plane equation, in contrast, accounts for the curvature and elasticity of the cable via an integral term.

A simplified model of *flexible inextensible cable* was then formulated. It was obtained by the axially deformable model by introducing an internal constraint expressing inextensibility. Following the *mixed formulation*, we stated the problem in terms of position (or displacement) vector and the stress vector, which is a *reactive force* playing the role of Lagrangian multiplier in the virtual power approach. Thus, the same balance equations holding for the extensible cable were derived, in which, however, the stretch is rigorously zero. These equations must only be sided by the inextensibility condition, since the constitutive law loses its meaning.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Chapter 6

Stiff Cables

In this chapter, we consider cables that are endowed with flexural and torsional stiffnesses. First, we discuss the technical problems in which such properties play a non-negligible role, so that they must necessarily be taken into account. Then, we formulate an approximated model of cable undergoing small curvatures and large extension, sufficiently simple, but able to account for new phenomena. To avoid cumbersome formulas, we restrict ourselves to planar natural configurations. Prestressed cables are successively addressed, for which nonlinear and linearized equations are derived, again, by assuming a planar reference configuration. Taut strings are considered as a particular case. Then, the order of magnitude of the terms of polar nature is evaluated, and reduced models are consistently derived. These models are unable to describe boundary layers, but capable to account for the twist of both non-shallow and shallow cables. Finally, inextensible stiff cables are considered, whose equations of motion are derived in the unprestressed and prestressed cases, as well as for reduced models.

6.1 Motivations

In Chapter 5 we studied cables as (perfectly) flexible bodies, modeled as a one-dimensional (1D) Cauchy continuum, embedded in a 3D-space. The choice of such a model was justified by an elastic energy analysis in which the cable was considered to be a slender beam, whose length-to-diameter ratio tends to infinity. However, it was stressed that such a model requires that the curvature radius assumed by the cable in the current configuration, both in bending and torsion, is sufficiently large with respect to the diameter of the cable cross-section. If this is *not* the case, i.e. if the cable

Mathematical Models of Beams and Cables Angelo Luongo and Daniele Zulli
© 2013 ISTE Ltd. Published 2013 by ISTE Ltd.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

undergoes large curvatures, the Cauchy model is inadequate to accurately describe the mechanical behavior of the body. In such problems, therefore, a richer model of cable equipped with flexural and torsional stiffnesses must be employed. We will refer to this as *stiff cable*, as opposed to flexible cable. Another possible name would be "cable-beam", to stress the double nature of the model.

Before going into the formulation, however, we want to mention some physical problems in which the use of a stiff cable is needed. The discussion will give us some hints for modeling.

Loss of pretension

The first drawback we can encounter in using the flexible cable model is the phenomenon of *loss of pretension*. This occurs, for example, in a suspended cable, weakly tensed by its own weight, when it experiences free (or forced) periodic oscillations, causing a dynamic stress that superimposes on the pretension. Since the dynamic contribution takes both signs in a period, if it is, in absolute value, smaller than the pretension (as happens for small-amplitude motions), then the cable remains tensed; if, in contrast, it is larger (as happens for sufficiently large motion amplitudes), the cable is compressed somewhere. Due to the fact that the flexural stiffness is small, the critical compressive load is also small, so that instability occurs via (perhaps local) buckling. Of course, such a phenomenon cannot be described by the flexible model, which is deprived of any flexural stiffness. Accounting for the latter, in contrast, it permits the transformation of the extensional energy into flexural energy [YOK 01], by also allowing the occurrence of *loops*, which have been observed in some studies concerning low-tension cables [GAT 02, GOY 07, GOY 05].

Boundary layers

When a cable is, for example, clamped at the ends, the flexible model is unable to satisfy the geometric boundary condition, since the order of the differential problem is too low. This drawback is *not* a consequence of nonlinearity, but manifests itself even in the linear range. For example, according to the flexible model, the free motion of a linear, planar, taut string is governed by a second-order differential equation in the transverse displacement only (the 1D-wave equation):

$$Tu'' = m\ddot{u} \qquad [6.1]$$

where T is the tension and m is the mass per unit length. Thus, the conditions of zero-displacement at the ends, $u_H = 0$, can be satisfied, but the conditions of zero-(macro-) rotation, $u'_H = 0$, cannot. If we add to the model a bending stiffness EJ, the relevant equation becomes of fourth order, namely:

$$EJu'''' + Tu'' = m\ddot{u} \qquad [6.2]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

so that the four conditions can now be satisfied. However, a new difficulty arises, since, due to the high slenderness of the cable, the highest derivative in the equation is multiplied by a small coefficient[1], i.e. equation [6.2] is a *singular differential equation* [BEN 99]. Problems like this are difficult to solve numerically, since the relevant models are *ill conditioned*. If one wants to solve the problem analytically, specific perturbation methods are available [BEN 99, NAY 73]. Without going into details, we can say that: (a) far from boundaries (*outer* regime) the response is weakly variable, so that the flexural effect can be neglected; (b) close to the ends (*inner* regime), in order to satisfy the boundary conditions, the response becomes strongly variable, entailing that the highest order derivative is large and the flexural effect must be taken into account.

It must be remarked that the flexural regime phenomenon is caused not only by the constraints at the boundaries, but it is also triggered by point-forces in the domain. These, indeed, would cause cusps in the flexible model, which are not kinematically admissible for the stiff model. Therefore, boundary layers manifest themselves in order to smooth the solution. As an example, when a force travels through a cable, a boundary layer moves with it.

Twist-depending forces

The external forces are often considered to be independent of the response of the structure. However, there are problems in which the interaction cannot be ignored. This is the case, for example, of the aerodynamic forces, which depend on the surface exposed to the flow. If the body is cylindrical, with a non-circular cross-section, and the flow is orthogonal to the axis of the cylinder, the aerodynamic forces depend on the rotation of the cylinder around its own axis. This circumstance roughly occurs when iced cables are invested by wind, possibly causing aeroelastic instability [BLE 01]. In such problems, the twist angle, although of minor importance in the description of cable mechanics, must mandatorily be accounted in the model, since it affects the external excitation.

An approximated model

Of course, an accurate formulation of the problem would require the use of the fully nonlinear model of curved beam that we illustrated in Chapter 3. This, however,

1. Equation [6.2], in non-dimensional form, becomes (hat omitted):

$$\epsilon^2 u'''' + u'' = \ddot{u} \qquad [6.3]$$

where $\hat{s} := s/l$, $\hat{t} := (t/l)\sqrt{T/m}$ are new coordinates, l is the string length, $\hat{u} := u/l$ is the non-dimensional displacement, and $\epsilon^2 := EJ/(Tl^2) \ll 1$ is a small parameter.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

is quite cumbersome and, when applied to cables, calls for overcoming the ill-conditioning of the equations. Therefore, for the purpose of this book, it is conjectured here that a *linear* approximation of the curvature of the beam, when combined with a *nonlinear* description of the unit extension of the flexible cable, would supply a reasonably simple model, amenable to an analytical treatment, and still able to capture the essential aspects of the mentioned phenomena. Of course, large curvatures, such as those analyzed in [YOK 01] and [GOY 05], cannot be dealt with this simplified model.

6.2 Unprestressed stiff cables

We formulate an approximated model of stiff cable that undergoes finite extensions and small curvatures. While, therefore, an exact kinematics is used to describe elongations of the cable, an infinitesimal kinematics is employed to describe flexural and torsional curvatures.

6.2.1 Kinematics

Let us consider the cable as a slender and *unshearable beam*, and model it as a 1D-polar continuum immersed in a 3D-space. Differently from the flexible cable, the stiff cable possesses a unique natural shape; we will take this as the reference configuration. Moreover, in order to simplify the model, we will make the following assumptions: (a) the cable is (initially) *planar*, and (b) the principal inertia triad coincides with the Frenet triad. However, we will allow the cable to change its original planar shape during the motion, by disposing itself on a spatial curve, along which the two triads are no longer coincident.

The displacement and rotation fields

Let the cable lie, at time $t = 0$, on the curve $\mathcal{S}$ of the plane π, having equation $\bar{\mathbf{x}} = \bar{\mathbf{x}}(s)$, where s is the arc length. A Frenet triad $\bar{\mathcal{B}}_f := (\bar{\mathbf{a}}_t(s), \bar{\mathbf{a}}_n(s), \bar{\mathbf{a}}_b(s))$ is taken along $\mathcal{S}$, with $\bar{\mathbf{a}}_t = \bar{\mathbf{x}}'$ the tangent to the curve, $\bar{\mathbf{a}}_n = \bar{\mathbf{x}}''/\bar{k}$ the (inward when $k > 0$) normal to the curve in π and $\bar{\mathbf{a}}_b$ the binormal orthogonal to π; here, $\bar{k} = \|\bar{\mathbf{x}}''\|$ is the (scalar) curvature of $\mathcal{S}$. Moreover, let us assume that the inertia principal triad of the cross-section, $\bar{\mathcal{B}} := (\bar{\mathbf{a}}_1(s), \bar{\mathbf{a}}_2(s), \bar{\mathbf{a}}_3(s))$, is aligned with the Frenet triad $\bar{\mathcal{B}}_f$, so that the deviation angle δ we considered in Chapter 3 is identically zero. Since the torsion of the planar curve is also zero, the Frenet formulas [5.6] reduce to:

$$\bar{\mathbf{a}}_1' = \bar{\kappa}\bar{\mathbf{a}}_2, \qquad \bar{\mathbf{a}}_2' = -\bar{\kappa}\bar{\mathbf{a}}_1, \qquad \bar{\mathbf{a}}_3' = 0 \qquad [6.4]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where we used $\bar{\mathbf{a}}_i$ $(i = 1, 2, 3)$ instead of $\bar{\mathbf{a}}_\alpha$ $(\alpha = t, n, b)$, and, moreover, we wrote $\bar{\kappa} := \bar{k}$, to keep a formal analogy with the curved beam (Chapter 3).

Let $\hat{\mathcal{S}}$ be the (generally) *spatial* curve on which the centerline lies at time t, of parametric equation $\mathbf{x} = \mathbf{x}(s, t)$, and let $\mathcal{B} := (\mathbf{a}_1(s,t), \mathbf{a}_2(s,t), \mathbf{a}_3(s,t))$ be the triad solid to the cross-section. If we assume that the cable is *shear-undeformable*, then $\mathbf{a}_1$ is tangent to $\hat{\mathcal{S}}$, but, in general, $\mathbf{a}_2, \mathbf{a}_3$ are no more aligned with the current normal and binormal.

The referential description of the motion requires assigning the vector field $\mathbf{u}(s,t)$ and the rotation tensor field $\mathbf{R}(s,t)$, defined by:

$$\mathbf{u} = \mathbf{x} - \bar{\mathbf{x}}, \qquad \mathbf{a}_i = \mathbf{R}\bar{\mathbf{a}}_i \tag{6.5}$$

where $\mathbf{R}$ depends on the three Tait–Bryan angles θ_i, as defined by equation [2.6]. Displacements and rotations are constrained by geometrical boundary conditions:

$$\mathbf{u}_H = \breve{\mathbf{u}}_H(t), \qquad \mathbf{R}_H = \breve{\mathbf{R}}_H(t) \tag{6.6}$$

REMARK 6.1. It should be noted that, differently from the flexible cable (Cauchy continuum), here (polar continuum) $\mathbf{R}$ is a *micro-rotation*. For this reason, we do not need to use the stretch λ in the definition of the current principal triad.

Strains and unshearability

We define the unit extension as for the flexible cable, namely $\mathbf{e} = e\bar{\mathbf{a}}_1$, with $e := \|\mathbf{x}'\| - 1$. Since $\mathbf{x}' = \bar{\mathbf{a}}_1 + \mathbf{u}'$, by letting $\mathbf{u} := \sum_{i=1}^{3} u_i \bar{\mathbf{a}}_i$ and using the Frenet formulas [6.4], we find:

$$e = \sqrt{\left(1 + u_1' - \bar{\kappa} u_2\right)^2 + \left(u_2' + \bar{\kappa} u_1\right)^2 + u_3'^2} - 1 \tag{6.7}$$

To express unshearability, we proceed as in section 4.3.2 (where we dealt with unshearable *straight* beams) by requiring that, in the current configuration, the normal to the cross-section $\mathbf{a}_1 = \mathbf{R}\bar{\mathbf{a}}_1$ coincides with the (unit) tangent to the centerline, $\mathbf{x}'/\|\mathbf{x}'\|$, i.e.:

$$\bar{\mathbf{a}}_1 + \mathbf{u}' = (1 + e)\,\mathbf{R}\bar{\mathbf{a}}_1 \tag{6.8}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

When this vector equation is projected onto the $\bar{\mathcal{B}}$-basis, it provides:

$$\begin{pmatrix} 1 + u_1' - \bar{\kappa} u_2 \\ u_2' + \bar{\kappa} u_1 \\ u_3' \end{pmatrix} = (1+e) \begin{pmatrix} \cos\theta_2 \cos\theta_3 \\ \cos\theta_2 \sin\theta_3 \\ -\sin\theta_2 \end{pmatrix} \quad [6.9]$$

from which, by eliminating e, the *slave* rotations θ_2, θ_3 can be determined as (exact) functions of the *master* displacements u_1, u_2, u_3[2]. With the aim, however, to formulate a simple model, we *linearize* these constraints, by taking $\cos\theta_i \simeq 1$ and $\sin\theta_i \simeq \theta_i$, and, consequently, we get:

$$\theta_2 = -u_3', \qquad \theta_3 = u_2' + \bar{\kappa} u_1 \quad [6.11]$$

With these internal constraints, the geometric boundary conditions [6.6] become:

$$\begin{aligned} & u_{1H} = \breve{u}_{1H}, \qquad u_{2H} = \breve{u}_{2H}, \quad u_{3H} = \breve{u}_{3H} \\ & \theta_{1H} = \breve{\theta}_{1H}, \quad -u_{3H}' = \breve{\theta}_{2H}, \quad u_{2H}' + \bar{\kappa} u_{1H} = \breve{\theta}_{3H} \end{aligned} \quad [6.12]$$

To evaluate the change of curvature, $\boldsymbol{\chi} := \sum_{i=1}^{3} \chi_i \bar{\mathbf{a}}_i$, we use equations [3.35], which we derived for the curved beam, and particularize them to the case of planar beam, for which $\bar{\kappa}_1 = 0$, $\bar{\kappa}_2 = 0$, $\bar{\kappa}_3 = \bar{\kappa}$. Under the assumption of small rotations, we obtain:

$$\chi_1 = \theta_1' - \bar{\kappa}\theta_2, \qquad \chi_2 = \theta_2' + \bar{\kappa}\theta_1, \qquad \chi_3 = \theta_3' \quad [6.13]$$

or, by using the constraints [6.11] and letting $\theta := \theta_1$:

$$\chi_1 = \theta' + \bar{\kappa} u_3', \qquad \chi_2 = -u_3'' + \bar{\kappa}\theta, \qquad \chi_3 = u_2'' + (\bar{\kappa} u_1)' \quad [6.14]$$

Velocity and spin

The velocity field is $\mathbf{v} := \dot{\mathbf{u}} = \sum_{i=1}^{3} \dot{u}_i \bar{\mathbf{a}}_i$. The spin $\boldsymbol{\omega} = \sum_{i=1}^{3} \omega_i \mathbf{a}_i$ has (current) components given by equation [2.62]. In the linear approximation for the rotations, they reduce to:

$$\omega_1 = \dot{\theta}_1, \qquad \omega_2 = \dot{\theta}_2, \qquad \omega_3 = \dot{\theta}_3 \quad [6.15]$$

2. Namely:

$$\tan\theta_2 = -\frac{u_3'}{\sqrt{(1 + u_1' - \bar{\kappa} u_2)^2 + (u_2' + \bar{\kappa} u_1)'^2}}, \quad \tan\theta_3 = \frac{u_2' + \bar{\kappa} u_1}{1 + u_1' - \bar{\kappa} u_2} \quad [6.10]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Strain-rates

Time-differentiation of the unit extension provides $\dot{\mathbf{e}} = \dot{e}\bar{\mathbf{a}}_1$, with:

$$\dot{e} = \frac{1}{1+e}\left[\left(1+u_1' - \bar{\kappa}u_2\right)\left(\dot{u}_1' - \bar{\kappa}\dot{u}_2\right) + \left(u_2' + \bar{\kappa}u_1\right)\left(\dot{u}_2' + \bar{\kappa}\dot{u}_1\right) + u_3'\dot{u}_3'\right] \quad [6.16]$$

Time-differentiation of the increment of curvature provides $\dot{\boldsymbol{\chi}} := \sum_{i=1}^{3} \dot{\chi}_i \bar{\mathbf{a}}_i$, where:

$$\dot{\chi}_1 = \dot{\theta}' + \bar{\kappa}\dot{u}_3', \qquad \dot{\chi}_2 = -\dot{u}_3'' + \bar{\kappa}\dot{\theta}, \qquad \dot{\chi}_3 = \dot{u}_2'' + \left(\bar{\kappa}\dot{u}_1\right)' \quad [6.17]$$

6.2.2 Dynamics

We consider the cable under the action of field-loads $\mathbf{p} = \sum_{i=1}^{3} \bar{p}_i \bar{\mathbf{a}}_i$ and end-loads $\mathbf{P}_H = \sum_{i=1}^{3} \bar{P}_{iH} \bar{\mathbf{a}}_{iH}$; moreover, only couples $\mathbf{c} = \bar{c}\bar{\mathbf{a}}_1$, $\mathbf{C}_H = \bar{C}_H \bar{\mathbf{a}}_{1H}$ are considered, for simplicity.

Balance equations

The balance equations are derived by the Virtual Power Principle (VPP), in the spirit of the displacement formulation (in which the slave variables are eliminated). The principle requires that:

$$\int_{\mathcal{S}} \left(\mathbf{R}^T\mathbf{t} \cdot \dot{\mathbf{e}} + \mathbf{R}^T\mathbf{m} \cdot \dot{\boldsymbol{\chi}}\right) ds = \int_{\mathcal{S}} \left(\mathbf{p} \cdot \mathbf{v} + \mathbf{c} \cdot \boldsymbol{\omega}\right) ds + \sum_{H=A}^{B} \left(\mathbf{P}_H \cdot \mathbf{v}_H + \mathbf{C}_H \cdot \boldsymbol{\omega}_H\right) \qquad \forall \mathbf{v}, \boldsymbol{\omega} \quad [6.18]$$

where $\mathbf{t} = T\mathbf{a}_1 + T_2\mathbf{a}_2 + T_3\mathbf{a}_3$ and $\mathbf{m} = M_1\mathbf{a}_1 + M_2\mathbf{a}_2 + M_3\mathbf{a}_3$ are the stresses, whose components, as usual, are expressed in the current basis. Here, T is the normal force or *tension* of the cable, M_1 is the torsional moment and M_2, M_3 are the bending moments. Moreover, T_2, T_3 are *shear forces* (note that they are *different from zero*); however, since $\dot{\mathbf{e}} = \dot{e}\bar{\mathbf{a}}_1$, they do not appear in the displacement formulation, being reactive stresses. By expanding the scalar products, the VPP becomes:

$$\int_{\mathcal{S}} \left(T\dot{e} + M_1\dot{\chi}_1 + M_2\dot{\chi}_2 + M_3\dot{\chi}_3\right) ds = \int_{\mathcal{S}} \left(\sum_{i=1}^{3} \bar{p}_i \dot{u}_i + \bar{c}\dot{\theta}\right) ds + \sum_{H=A}^{B} \left[\sum_{i=1}^{3} \bar{P}_i \dot{u}_i + \bar{C}\dot{\theta}\right]_H \qquad \forall \left(\dot{u}_1, \dot{u}_2, \dot{u}_3, \dot{\theta}\right) \quad [6.19]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Concerning the virtual work spent by the tension, by using the strain-rate-velocity relationships [6.16] and [6.17] and integrating by parts, we have:

$$\begin{aligned}\int_{\mathcal{S}} T\dot{e}ds = &\int_{\mathcal{S}} \left[-\left(\frac{T}{1+e}\left(1+u_1'-\bar{\kappa}u_2\right)\right)'\right.\\&\left.+\bar{\kappa}\left(\frac{T}{1+e}\left(u_2'+\bar{\kappa}u_1\right)\right)\dot{u}_1\right]ds\\&+\int_{\mathcal{S}}\left[-\left(\frac{T}{1+e}\left(u_2'+\bar{\kappa}u_1\right)\right)'-\bar{\kappa}\left(\frac{T}{1+e}\left(1+u_1'-\bar{\kappa}u_2\right)\right)\right]\dot{u}_2ds\\&+\int_{\mathcal{S}}-\left(\frac{T}{1+e}u_3'\right)'\dot{u}_3ds\\&+\left[\frac{T}{1+e}\left(1+u_1'-\bar{\kappa}u_2\right)\dot{u}_1+\frac{T}{1+e}\left(u_2'+\bar{\kappa}u_1\right)\dot{u}_2+\frac{T}{1+e}u_3'\dot{u}_3\right]_A^B\end{aligned} \quad [6.20]$$

Similarly:

$$\begin{aligned}&\int_{\mathcal{S}}\left(M_1\dot{\chi}_1+M_2\dot{\chi}_2+M_3\dot{\chi}_3\right)ds\\&=\int_{\mathcal{S}}\left[\left(-M_1'+\bar{\kappa}M_2\right)\dot{\theta}-\bar{\kappa}M_3'\dot{u}_1+M_3''\dot{u}_2+\left(-M_2''-\left(\bar{\kappa}M_1\right)'\right)\dot{u}_3\right]ds\\&+\left[M_1\dot{\theta}+\bar{\kappa}M_3\dot{u}_1+M_3\dot{u}_2'-M_3'\dot{u}_2-M_2\dot{u}_3'+\left(M_2'+\bar{\kappa}M_1\right)\dot{u}_3\right]_A^B\end{aligned} \quad [6.21]$$

The VPP, therefore, provides the following field equations:

$$\begin{aligned}&\bar{\kappa}M_3'+\left[\frac{T}{1+e}\left(1+u_1'-\bar{\kappa}u_2\right)\right]'-\bar{\kappa}\frac{T}{1+e}\left(u_2'+\bar{\kappa}u_1\right)+\bar{p}_1=m\ddot{u}_1\\&-M_3''+\left[\frac{T}{1+e}\left(u_2'+\bar{\kappa}u_1\right)\right]'+\bar{\kappa}\frac{T}{1+e}\left(1+u_1'-\bar{\kappa}u_2\right)+\bar{p}_2=m\ddot{u}_2\\&M_2''+\left(\bar{\kappa}M_1\right)'+\left(\frac{T}{1+e}u_3'\right)'+\bar{p}_3=m\ddot{u}_3\\&M_1'-\bar{\kappa}M_2+\bar{c}=I\ddot{\theta}\end{aligned} \quad [6.22]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and the mechanical boundary conditions:

$$\begin{aligned}
&\left[\bar{P}_1 \pm \left(\bar{\kappa} M_3 + \frac{T}{1+e}\left(1 + u_1' - \bar{\kappa} u_2\right)\right)\right]_H \dot{u}_{1H} = 0 \\
&\left[\bar{P}_2 \pm \left(-M_3' + \frac{T}{1+e}\left(u_2' + \bar{\kappa} u_1\right)\right)\right]_H \dot{u}_{2H} = 0 \\
&\left[\pm M_3\right]_H \dot{u}_{2H}' = 0 \\
&\left[\bar{P}_3 \pm \left(M_2' + \bar{\kappa} M_1 + \frac{T}{1+e} u_3'\right)\right]_H \dot{u}_{3H} = 0 \\
&\left[\mp M_2\right]_H \dot{u}_{3H}' = 0 \\
&\left[\bar{C} \pm M_1\right]_H \dot{\theta}_H = 0
\end{aligned} \qquad [6.23]$$

where inertia forces have been added in the field, m being the linear mass density and $I := I_1$ the mass inertia moment with respect to $\mathbf{a}_1$.

6.2.3 The elastic law

The *active* stresses are related to the admissible strains by the constitutive law. By assuming that the material is linearly hyperelastic and there is no prestress, the law becomes:

$$T = EAe, \qquad M_1 = GJ_1\chi_1, \qquad M_2 = EJ_2\chi_2, \qquad M_3 = EJ_3\chi_3 \qquad [6.24]$$

where EA is the axial stiffness, GJ_1 is the torsional stiffness and $EJ_2,\ EJ_3$ are the flexural stiffnesses.

6.2.4 The Fundamental Problem

The Fundamental Problem is governed by four strain–displacement relationships [6.7] and [6.14]; four balance equations [6.22]; four elastic laws [6.24] and the geometrical/mechanical boundary conditions [6.12] and [6.23]. There are 12 unknowns: four displacements, four strains and four active stresses.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

When these equations are combined only in terms of displacements, we obtain the equations of motion:

$$
\begin{aligned}
&EJ_3\bar{\kappa}\left(u_2'+\bar{\kappa}u_1\right)''+\left[\frac{EAe}{1+e}\left(1+u_1'-\bar{\kappa}u_2\right)\right]'-\bar{\kappa}\left[\frac{EAe}{1+e}\left(u_2'+\bar{\kappa}u_1\right)\right]\\
&\qquad+\bar{p}_1=m\ddot{u}_1\\
&-EJ_3\left(u_2'+\bar{\kappa}u_1\right)'''+\left[\frac{EAe}{1+e}\left(u_2'+\bar{\kappa}u_1\right)\right]'+\bar{\kappa}\left[\frac{EAe}{1+e}\left(1+u_1'-\bar{\kappa}u_2\right)\right]\\
&\qquad+\bar{p}_2=m\ddot{u}_2\\
&EJ_2\left(-u_3''+\bar{\kappa}\theta\right)''+GJ_1\left[\bar{\kappa}\left(\theta'+\bar{\kappa}u_3'\right)\right]'+\left(\frac{EAe}{1+e}u_3'\right)'+\bar{p}_3=m\ddot{u}_3\\
&GJ_1\left(\theta'+\bar{\kappa}u_3'\right)'-EJ_2\bar{\kappa}\left(-u_3''+\bar{\kappa}\theta\right)+\bar{c}=I\ddot{\theta}
\end{aligned}
\qquad [6.25]
$$

with the mechanical boundary conditions:

$$
\begin{aligned}
&\mp\left[\bar{\kappa}EJ_3\left(u_2''+(\bar{\kappa}u_1)'\right)+\frac{EAe}{1+e}\left(1+u_1'-\bar{\kappa}u_2\right)\right]_H=\bar{P}_{1H}\\
&\mp\left[-EJ_3\left(u_2'''+(\bar{\kappa}u_1)''\right)+\frac{EAe}{1+e}\left(u_2'+\bar{\kappa}u_1\right)\right]_H=\bar{P}_{2H}\\
&\mp\left[EJ_3\left(u_2''+(\bar{\kappa}u_1)'\right)\right]_H=0\\
&\mp\left[EJ_2\left(-u_3'''+(\bar{\kappa}\theta)'\right)+\bar{k}GJ_1\left(\theta'+\bar{\kappa}u_3'\right)+\frac{EAe}{1+e}u_3'\right]_H=\bar{P}_{3H}\\
&\pm\left[EJ_2\left(-u_3''+\bar{\kappa}\theta\right)\right]_H=0\\
&\mp\left[GJ_1\left(\theta'+\bar{\kappa}u_3'\right)\right]_H=\bar{C}_H
\end{aligned}
\qquad [6.26]
$$

and/or geometrical boundary conditions [6.12].

REMARK 6.2. The first three equations of motion are fully coupled by the nonlinear unit strain. The fourth equation, due to linearization of the curvatures, is linear, and, due to the planarity of the cable, couples twist and out-of-plane displacements only. However, due to the mentioned extensional coupling, twist triggers the in-plane displacements too.

6.3 Prestressed stiff cables

As for the flexible cable (section 5.3), we are interested in describing the motion of stiff cables that, in their reference configuration, are *prestressed by static forces*.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Accordingly, we will measure displacements and strains starting from this configuration, rather than from the natural configuration. We will assume that a preload analysis, leading the cable from the natural to the reference configuration, has already been performed via the model of the previous section, and we want to describe the response to the cable to incremental loads, possibly of dynamic type. Both the nonlinear and the linearized theories are addressed for the cable, and then specialized to taut strings.

6.3.1 Nonlinear model

Pre-existing equilibrium state

We assume that *the prestressed cable still keeps its natural planar shape*. This is rigorously true if the forces belong to the same plane (and couples are absent), while it must be taken as a further approximation if this is not the case. For example, a cable under self-weight and horizontal static wind forces only approximately disposes itself on a plane inclined on the vertical axis. Moreover, aerodynamic couples distributed along the centerline also contribute to the loss of planarity. As we already commented for beams, it is quite commonly accepted to account for a state of prestress in the body by neglecting the change of geometry, thus confusing the natural and the prestressed configurations. This hypothesis, however, must be taken with caution in a cable, which is a much more flexible body than a beam.

With this clarification, we consider *preloads* $\mathring{\mathbf{p}}, \mathring{\mathbf{c}}, \mathring{\mathbf{P}}_H, \mathring{\mathbf{C}}_H$ in equilibrium with the *prestresses* $\mathring{\mathbf{t}}, \mathring{\mathbf{m}}$. They all satisfy the static version of equations [6.22] and [6.23], when these are written in the reference configuration, i.e. for zero displacements and strains, namely:

$$\begin{aligned} &\bar{\kappa}\mathring{M}_3' + \mathring{T}' + \mathring{p}_t = 0 \\ -&\mathring{M}_3'' + \bar{\kappa}\mathring{T} + \mathring{p}_n = 0 \\ &\mathring{M}_2'' + \left(\bar{\kappa}\mathring{M}_1\right)' + \mathring{p}_3 = 0 \\ &\mathring{M}_1' - \bar{\kappa}\mathring{M}_2 + \mathring{c} = 0 \end{aligned} \qquad [6.27]$$

and:

$$\begin{aligned} &\mp\left[\bar{k}\mathring{M}_3 + \mathring{T}\right]_H = \mathring{P}_{1H}, \quad \mp\left[-\mathring{M}_3'\right]_H = \mathring{P}_{2H}, \quad \mp\left[\mathring{M}_3\right]_H = 0 \\ &\mp\left[\mathring{M}_2' + \bar{k}\mathring{M}_1\right]_H = \mathring{P}_{3H}, \quad \pm\left[\mathring{M}_2\right]_H = 0, \quad \mp\left[\mathring{M}_1\right]_H = \mathring{C}_H \end{aligned} \qquad [6.28]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Incremental balance equations

When dynamic *incremental loads* $\tilde{\mathbf{p}}, \tilde{\mathbf{c}}, \tilde{\mathbf{P}}_H, \tilde{\mathbf{C}}_H$ take action on the prestressed cable, the balance equations [6.22] and [6.23] must be satisfied, with $\mathbf{p}, \mathbf{c}, \mathbf{P}_H, \mathbf{C}_H$ being the *total loads* (namely $\mathbf{p} := \mathring{\mathbf{p}} + \tilde{\mathbf{p}}$ and similar). By subtracting from these equations the equilibrium equations [6.27] and [6.28], holding in the prestressed state, the incremental equations are derived. The latter, however, assume a simple form, only if we neglect the unit strain with respect to 1, as we observed for flexible cables in formulating a quasi-exact model (section 5.3.1)[3]. By proceeding in this way and denoting with $\tilde{T} := T - \mathring{T}$, $\tilde{M}_i := M_i - \mathring{M}_i$ the increments of stress, we obtain:

$$\begin{aligned}
&\bar{\kappa}\tilde{M}_3' + \tilde{T}' + \left[T\left(u_1' - \bar{\kappa}u_2\right)\right]' - \bar{\kappa}T\left(u_2' + \bar{\kappa}u_1\right) + \tilde{p}_1 = m\ddot{u}_1 \\
&-\tilde{M}_3'' + \bar{\kappa}\tilde{T} + \left[T\left(u_2' + \bar{\kappa}u_1\right)\right]' + \bar{\kappa}T\left(u_1' - \bar{\kappa}u_2\right) + \tilde{p}_2 = m\ddot{u}_2 \\
&\tilde{M}_2'' + \left[\bar{\kappa}\tilde{M}_1\right]' + \left(Tu_3'\right)' + \tilde{p}_3 = m\ddot{u}_3 \\
&\tilde{M}_1' - \bar{\kappa}\tilde{M}_2 + \tilde{c} = I\ddot{\theta}
\end{aligned} \qquad [6.29]$$

together with:

$$\begin{aligned}
&\mp\left[\bar{\kappa}\tilde{M}_3 + \tilde{T} + T\left(u_1' - \bar{\kappa}u_2\right)\right]_H = \tilde{P}_{1H} \\
&\mp\left[-\tilde{M}_3' + T\left(u_2' + \bar{\kappa}u_1\right)\right]_H = \tilde{P}_{2H} \\
&\mp\left[\tilde{M}_3\right]_H = 0 \\
&\mp\left[\tilde{M}_2' + \bar{\kappa}\tilde{M}_1 + Tu_3'\right]_H = \tilde{P}_{3H} \\
&\pm\left[\tilde{M}_2\right]_H = 0 \\
&\mp\left[\tilde{M}_1\right]_H = \tilde{C}_H
\end{aligned} \qquad [6.30]$$

Note that these equations reduce to the incremental balance equations for the flexible cable (equations [5.76] and [5.77] with torsion vanished), if moments are ignored.

3. For example, in the first field equation, a term like this appears:

$$\left[\frac{T}{1+e}\left(1 + u_t' - \bar{k}u_n\right)\right]' - \mathring{T}' \simeq \left[\tilde{T} + T\left(u_t' - \bar{k}u_n\right)\right]'$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Elastic law

Due to the prestress, the linear elastic law is non-homogeneous, namely:

$$
\begin{aligned}
T &= \overset{\circ}{T} + EAe, & M_1 &= \overset{\circ}{M}_1 + GJ_1\chi_1 \\
M_2 &= \overset{\circ}{M}_2 + EJ_2\chi_2, & M_3 &= \overset{\circ}{M}_3 + EJ_3\chi_3
\end{aligned}
\qquad [6.31]
$$

However, since only the increment of moments appears in the balance equations (due to linearization of the curvatures), the elastic law becomes:

$$
T = \overset{\circ}{T} + EAe, \quad \tilde{M}_1 = GJ_1\chi_1, \quad \tilde{M}_2 = EJ_2\chi_2, \quad \tilde{M}_3 = EJ_3\chi_3 \qquad [6.32]
$$

The Fundamental Problem

The incremental form of the Fundamental Problem is constituted by the strain-displacement relationships (equations [6.7] and [6.14]), the balance equations [6.29], the elastic law [6.32], and the boundary conditions ([6.6] and [6.30]). When the problem is stated in terms of displacements only, the following equations of motion are derived in the field:

$$
\begin{aligned}
&\bar{\kappa}EJ_3\left(u_2' + \bar{\kappa}u_1\right)'' + EAe' + \left[\left(\overset{\circ}{T} + EAe\right)\left(u_1' - \bar{\kappa}u_2\right)\right]' \\
&\qquad - \bar{\kappa}\left(\overset{\circ}{T} + EAe\right)\left(u_2' + \bar{\kappa}u_1\right) + \tilde{p}_1 = m\ddot{u}_1 \\
&-EJ_3\left(u_2' + \bar{\kappa}u_1\right)''' + \bar{\kappa}EAe + \left[\left(\overset{\circ}{T} + EAe\right)\left(u_2' + \bar{\kappa}u_1\right)\right]' \\
&\qquad + \bar{\kappa}\left(\overset{\circ}{T} + EAe\right)\left(u_1' - \bar{\kappa}u_2\right) + \tilde{p}_2 = m\ddot{u}_2 \\
&EJ_2\left(-u_3'' + \bar{\kappa}\theta\right)'' + \left[\bar{\kappa}GJ_1\left(\theta' + \bar{\kappa}u_3'\right)\right]' \\
&\qquad + \left[\left(\overset{\circ}{T} + EAe\right)u_3'\right]' + \tilde{p}_3 = m\ddot{u}_3 \\
&GJ_1\left(\theta' + \bar{\kappa}u_3'\right)' - \bar{\kappa}EJ_2\left(-u_3'' + \bar{\kappa}\theta\right) + \tilde{c} = I\ddot{\theta}
\end{aligned}
\qquad [6.33]
$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

to be put beside, on the boundary, by equation [6.6] and:

$$
\begin{aligned}
&\mp \left[\bar{\kappa} E J_3 \left(u_2' + \bar{\kappa} u_1\right)' + EAe + \left(\mathring{T} + EAe\right)\left(u_1' - \bar{\kappa} u_2\right)\right]_H = \tilde{P}_{1H} \\
&\mp \left[-E J_3 \left(u_2' + \bar{\kappa} u_1\right)'' + \left(\mathring{T} + EAe\right)\left(u_2' + \bar{\kappa} u_1\right)\right]_H = \tilde{P}_{2H} \\
&\mp \left[E J_3 \left(u_2' + \bar{\kappa} u_1\right)'\right]_H = 0 \\
&\mp \left[E J_2 \left(-u_3'' + \bar{\kappa}\theta\right)' + \bar{k} G J_1 \left(\theta' + \bar{\kappa} u_3'\right) + \left(\mathring{T} + EAe\right) u_3'\right]_H = \tilde{P}_{3H} \\
&\pm \left[E J_2 \left(-u_3'' + \bar{\kappa}\theta\right)\right]_H = 0 \\
&\mp \left[G J_1 \left(\theta' + \bar{\kappa} u_3'\right)\right]_H = \tilde{C}_H
\end{aligned}
\qquad [6.34]
$$

where the unit strain e is expressed by equation [6.7].

6.3.2 The linearized model

If we are interested in *small* motions around the prestressed configuration, we can linearize the incremental problem. By linearizing equation [6.7] and appending (already linear) equation [6.14], the strain–displacement relationships become:

$$
e = u_1' - \bar{\kappa} u_2, \qquad \chi_1 = \theta' + \bar{\kappa} u_3', \qquad \chi_2 = -u_3'' + \bar{\kappa}\theta, \qquad \chi_3 = u_2'' + (\bar{\kappa} u_1)' \qquad [6.35]
$$

Linearization of the incremental balance equations [6.29] and [6.30] is simply obtained by replacing T by $\mathring{T}$, namely:

$$
\begin{aligned}
&\bar{\kappa}\tilde{M}_3' + \tilde{T}' + \left[\mathring{T}\left(u_1' - \bar{\kappa} u_2\right)\right]' - \bar{\kappa}\mathring{T}\left(u_2' + \bar{\kappa} u_1\right) + \tilde{p}_1 = m\ddot{u}_1 \\
&-\tilde{M}_3'' + \bar{\kappa}\tilde{T} + \left[\mathring{T}\left(u_2' + \bar{\kappa} u_1\right)\right]' + \bar{\kappa}\mathring{T}\left(u_1' - \bar{\kappa} u_2\right) + \tilde{p}_2 = m\ddot{u}_2 \\
&\tilde{M}_2'' + \left(\bar{\kappa}\tilde{M}_1\right)' + \left(\mathring{T} u_3'\right)' + \tilde{p}_3 = m\ddot{u}_3 \\
&\tilde{M}_1' - \bar{\kappa}\tilde{M}_2 + \tilde{c} = I\ddot{\theta}
\end{aligned}
\qquad [6.36]
$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and:

$$\begin{aligned}
&\mp\left[\bar{k}\tilde{M}_3+\tilde{T}+\mathring{T}\left(u_1'-\bar{\kappa}u_2\right)\right]_H=\tilde{P}_{1H}\\
&\mp\left[-\tilde{M}_3'+\mathring{T}\left(u_2'+\bar{\kappa}u_1\right)\right]_H=\tilde{P}_{2H}\\
&\mp\left[\tilde{M}_3\right]_H=0\\
&\mp\left[\tilde{M}_2'+\bar{\kappa}\tilde{M}_1+\mathring{T}u_3'\right]_H=\tilde{P}_{3H}\\
&\pm\left[\tilde{M}_2\right]_H=0\\
&\mp\left[\tilde{M}_1\right]_H=\tilde{C}_H
\end{aligned}\qquad [6.37]$$

The incremental constitutive law follows from equation [6.32]:

$$\tilde{T}=EAe,\qquad \tilde{M}_1=GJ_1\chi_1,\qquad \tilde{M}_2=EJ_2\chi_2,\qquad \tilde{M}_3=EJ_3\chi_3\qquad [6.38]$$

Combination of these equations leads to the linearized equations of motion[4]:

$$\begin{aligned}
&\bar{\kappa}EJ_3\left(u_2'+\bar{\kappa}u_1\right)''+EA\left(u_1'-\bar{\kappa}u_2\right)'+\left[\mathring{T}\left(u_1'-\bar{\kappa}u_2\right)\right]'\\
&\qquad-\bar{\kappa}\mathring{T}\left(u_2'+\bar{\kappa}u_1\right)+\tilde{p}_1=m\ddot{u}_1\\
&-EJ_3\left(u_2'+\bar{\kappa}u_1\right)'''+\bar{\kappa}EA\left(u_1'-\bar{k}u_2\right)\\
&\qquad+\left[\mathring{T}\left(u_2'+\bar{\kappa}u_1\right)\right]'+\bar{k}\mathring{T}\left(u_1'-\bar{\kappa}u_2\right)+\tilde{p}_2=m\ddot{u}_2\\
&EJ_2\left(-u_3''+\bar{\kappa}\theta\right)''+\left[\bar{k}GJ_1\left(\theta'+\bar{\kappa}u_3'\right)\right]'+\left[\mathring{T}u_3'\right]'+\tilde{p}_3=m\ddot{u}_3\\
&GJ_1\left(\theta'+\bar{\kappa}u_3'\right)'-\bar{k}EJ_2\left(-u_3''+\bar{\kappa}\theta\right)+\tilde{c}=I\ddot{\theta}
\end{aligned}\qquad [6.39]$$

4. These equations are of type $\boldsymbol{L}\boldsymbol{w}+\boldsymbol{G}\boldsymbol{w}=\tilde{\boldsymbol{p}}$, in the field, and $\boldsymbol{\mathcal{L}}_H\boldsymbol{w}+\boldsymbol{\mathcal{G}}_H\boldsymbol{w}=\tilde{\boldsymbol{P}}$, on the boundary, as we saw in Chapter 1.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and the mechanical boundary conditions:

$$\begin{aligned}
&\mp \left[\bar{\kappa} E J_3 \left(u_3' + \bar{\kappa} u_1\right)' + EA \left(u_1' - \bar{\kappa} u_2\right) + \overset{\circ}{T} \left(u_1' - \bar{\kappa} u_2\right)\right]_H = \bar{P}_{1H} \\
&\mp \left[-E J_3 \left(u_2' + \bar{\kappa} u_1\right)'' + \overset{\circ}{T} \left(u_2' + \bar{\kappa} u_1\right)\right]_H = \bar{P}_{2H} \\
&\mp \left[E J_3 \left(u_2' + \bar{\kappa} u_1\right)'\right]_H = 0 \\
&\mp \left[E J_2 \left(-u_3'' + \bar{\kappa}\theta\right)' + \bar{k} G J_1 \left(\theta' + \bar{\kappa} u_3'\right) + \overset{\circ}{T} u_3'\right]_H = \bar{P}_{3H} \\
&\pm \left[E J_2 \left(-u_3'' + \bar{\kappa}\theta\right)\right]_H = 0 \\
&\mp \left[G J_1 \left(\theta' + \bar{\kappa} u_3'\right)\right]_H = \bar{C}_H
\end{aligned} \qquad [6.40]$$

to be sided by the geometric boundary conditions. These equations reduce to those for the planar flexible model (equations [5.89] and [5.90], with $\bar{\tau} = 0$) when the flexural and torsional contributions are neglected.

6.3.3 Taut strings

We now consider a stiff cable that, in its natural configuration, is rectilinear and aligned along the $\mathbf{i}_1$-axis of the Cartesian basis $\mathcal{B}_e := (\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$, coincident with the intrinsic one. Let the cable be solicited exclusively by two equal and opposite static forces $-\overset{\circ}{\mathbf{P}}_A = \overset{\circ}{\mathbf{P}}_B =: \overset{\circ}{T}\mathbf{i}_1$. Since $\bar{\kappa} = 0$, distributed loads are zero (i.e. $\overset{\circ}{p}_1 = \overset{\circ}{p}_2 = \overset{\circ}{p}_3 = \overset{\circ}{c} = 0$), and transverse end-forces and end-couples are zero too (i.e. $\overset{\circ}{P}_{2H} = \overset{\circ}{P}_{3H} = \overset{\circ}{C}_H = 0$), then the equilibrium equations [6.27] and [6.28], relevant to the prestressed configuration, admit the solution:

$$\overset{\circ}{T} = \text{const}, \qquad \overset{\circ}{M}_1 = \overset{\circ}{M}_2 = \overset{\circ}{M}_3 = 0 \qquad [6.41]$$

i.e. the cable is taut with zero bending and torsional moments, as in the flexible case (section 5.3.3).

Nonlinear model

Since the cable is rectilinear ($\bar{\kappa} = 0$), the strains [6.7] and [6.14] reduce to:

$$e = \sqrt{\left(1 + u_1'\right)^2 + u_2'^2 + u_3'^2} - 1 \qquad [6.42]$$

and:

$$\chi_1 = \theta', \qquad \chi_2 = -u_3'', \qquad \chi_3 = u_2'' \qquad [6.43]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The incremental balance equations [6.33] and [6.34] also simplify into:

$$\begin{aligned} &\tilde{T}' + (Tu_1')' + \tilde{p}_1 = m\ddot{u}_1 \\ -&\tilde{M}_3'' + (Tu_2')' + \tilde{p}_2 = m\ddot{u}_2 \\ &\tilde{M}_2'' + (Tu_3')' + \tilde{p}_3 = m\ddot{u}_3 \\ &\tilde{M}_1' + \tilde{c} = I\ddot{\theta} \end{aligned} \quad [6.44]$$

and:

$$\begin{aligned} &\mp \left[\tilde{T} + Tu_1'\right]_H = \tilde{P}_{1H} \\ &\mp \left[-\tilde{M}_3' + Tu_2'\right]_H = \tilde{P}_{2H} \\ &\mp \left[\tilde{M}_3\right]_H = 0 \\ &\mp \left[\tilde{M}_2' + Tu_3'\right]_H = \tilde{P}_{3H} \\ &\pm \left[\tilde{M}_2\right]_H = 0 \\ &\mp \left[\tilde{M}_1\right]_H = \tilde{C}_H \end{aligned} \quad [6.45]$$

The elastic law [6.31], by accounting for the prestresses [6.41], becomes:

$$T = \mathring{T} + EAe, \qquad M_1 = GJ_1\chi_1, \qquad M_2 = EJ_2\chi_2, \qquad M_3 = EJ_3\chi_3 \quad [6.46]$$

The Fundamental Problem, when formulated in terms of displacements only, becomes (compare it with equations [5.95] and [5.97], relevant to the flexible model):

$$\begin{aligned} &\mathring{T}u_1'' + [EAe\,(1 + u_1')]' + \tilde{p}_1 = m\ddot{u}_1 \\ -&EJ_3u_2'''' + \mathring{T}u_2'' + (EAeu_2')' + \tilde{p}_2 = m\ddot{u}_2 \\ -&EJ_2u_3'''' + \mathring{T}u_3'' + (EAeu_3')' + \tilde{p}_3 = m\ddot{u}_3 \\ &GJ_1\theta'' + \tilde{c} = I\ddot{\theta} \end{aligned} \quad [6.47]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and:

$$\begin{aligned}
&\mp\left[\mathring{T}u_1' + EAe\left(1+u_1'\right)\right]_H = \tilde{P}_{1H}\\
&\mp\left[-EJ_3u_2''' + \left(\mathring{T} + EAe\right)u_2'\right]_H = \tilde{P}_{2H}\\
&\mp\left[EJ_3u_2''\right]_H = 0\\
&\mp\left[-EJ_2u_3''' + \left(\mathring{T} + EAe\right)u_3'\right]_H = \tilde{P}_{3H}\\
&\pm\left[-EJ_2u_3''\right]_H = 0\\
&\mp\left[GJ_1\theta'\right]_H = \tilde{C}_H
\end{aligned} \qquad [6.48]$$

plus geometric boundary conditions.

REMARK 6.3. The field equation and boundary conditions governing the torsional motion of the taut string are uncoupled from the remaining ones. Since curvatures have been linearized, the governing equation is linear.

Linearized model

When the previous equations are linearized around the prestressed configuration, i.e. small motions around it are considered, the problem becomes:

$$\begin{aligned}
&\left(EA + \mathring{T}\right)u'' + \tilde{p}_1 = m\ddot{u}_1\\
&-EJ_3u_2'''' + \mathring{T}u_2'' + \tilde{p}_2 = m\ddot{u}_2\\
&-EJ_2u_3'''' + \mathring{T}u_3'' + \tilde{p}_3 = m\ddot{u}_3\\
&GJ_1\theta'' + \tilde{c} = I\ddot{\theta}
\end{aligned} \qquad [6.49]$$

and:

$$\begin{aligned}
&\mp\left[\left(EA + \mathring{T}\right)u_1'\right]_H = \tilde{P}_{1H}\\
&\mp\left[-EJ_3u_2''' + \mathring{T}u_2'\right]_H = \tilde{P}_{2H}\\
&\mp\left[EJ_3u_2''\right]_H = 0\\
&\mp\left[-EJ_2u_3''' + \mathring{T}u_3'\right]_H = \tilde{P}_{3H}\\
&\pm\left[-EJ_2u_3''\right]_H = 0\\
&\mp\left[GJ_1\theta'\right]_H = \tilde{C}_H
\end{aligned} \qquad [6.50]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where the linearized strain $e = u_1'$ has been used. Geometric boundary conditions must also be enforced. These equations reduce to those for the flexible model, when flexure and torsion are ignored (section 5.3.3); as in that case, all the equations are uncoupled.

6.4 Reduced models

The large-extension small-curvature model of stiff cable (equations [6.33] and [6.34]), incorporates the flexible model, in the sense that: (a) the linear momentum equations contain *additional terms* due to the bending and torsion of the unshearable beam; (b) the angular momentum equation around the tangent is *a further balance equation*. This disappears when the twist angle is not included in the set of the kinematic descriptors. We will now investigate the role of these differences with respect to the flexible model, in order to understand *if* and *when* they can be neglected, to get a consistently *reduced model*. To acheive this goal, we will perform an order of magnitude analysis, by distinguishing the cases of *sagged* and *shallow* cables.

6.4.1 Sagged cables

If the cable is sagged, we can assume that its radius of curvature in the reference configuration is of the order of the length of the cable (this entailing that the sag-to-length ratio is of order 1); therefore, $\bar{\kappa} = \mathrm{O}\left(l^{-1}\right)$. Moreover, we assume that all the translation components are of the same order, i.e. $u_i = \mathrm{O}(u)$, $(i = 1, 2, 3)$, where $u := \|\mathbf{u}\|$ is a measure of the amplitude. If we are interested in motions in which the displacements vary on a characteristic length of the order of l (as it happens for the first few natural modes of the cable), we can assume that $\partial^n u_i / \partial s^n = \mathrm{O}(u/l^n)$. Similarly, we assume that $\partial^n \theta / \partial s^n = \mathrm{O}(\theta/l^n)$, but we still have to estimate θ and to relate it to u, which is a less easy task. To this end, we consider the homogeneous static version of equation [6.33d] and consider u_3 as a "know-term". Since $GJ_1/EJ_2 = \mathrm{O}(1)$, by referring to an average (constant) curvature $\bar{\kappa}$, we rewrite this equation as:

$$\theta'' - \bar{\kappa}^2 \theta = \mathrm{O}(\bar{\kappa} u_3'') \qquad [6.51]$$

The first two terms on the left side, by virtue of the previous assumptions, are of the same order of magnitude. By balancing them with the right side, we get $\theta = \mathrm{O}(u/l)$, i.e. the estimation we sought for.

With these results we note that each increment of curvature [6.14] is made up of two terms of equal order, and that these orders are all equal among them, namely $\chi_i = \mathrm{O}\left(u/l^2\right)$, $(i = 1, 2, 3)$. Therefore, forces due to bending and torsion, which

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

appear in the equations of motion [6.33], are of the same order, and no simplification is allowed among them. However, if we compare these flexural/torsional effects with the extensional or geometric effects, we discover that they are much smaller. As a matter of fact, in equation [6.33a], we have:

$$\frac{\bar{\kappa} E J_3 u_2'''}{\mathring{T} u_1''} = \mathrm{O}\left(\frac{E J_3}{\mathring{T} l^2}\right) = \mathrm{O}\left(\frac{E}{\mathring{\sigma}} \frac{\rho_3^2}{l^2}\right) \ll 1 \qquad [6.52]$$

where $\mathring{\sigma} := \mathring{T}/A$ and $\rho_3^2 := J_3/A$[5]. Similarly, in the equation [6.33c] (and [6.33b]), we have:

$$\mathrm{O}\left(\frac{E J_2 u_3''''}{\mathring{T} u_3''}\right) = \mathrm{O}\left(\frac{\bar{\kappa} G J_1 \theta''}{\mathring{T} u_3''}\right) = \mathrm{O}\left(\frac{E}{\mathring{\sigma}} \frac{\rho_2^2}{l^2}\right) \ll 1 \qquad [6.53]$$

This analysis is consistent with the discussion at the beginning of this chapter. The flexural and torsional terms appearing in the linear momentum equations are important only near the constraints, where boundary layers occur. Here, in fact, the derivatives of the displacements are much larger than far from the constraints, e.g. $u_2' \gg \mathrm{O}\,(u/l)$, so that the previous analysis, valid far from the constraints, no longer holds. If, however, we are not interested in investigating boundary layers, but we accept a local error, *we can neglect all flexural and torsional terms in equations [6.33a,b,c]*, which therefore reduce to those for the flexible model. However, equation [6.33d], which expresses the tangent angular momentum balance equation, survives, since all terms are of the same order. Therefore, the reduced model becomes:

$$\begin{aligned}
&EAe' + \left[\left(\mathring{T} + EAe\right)(u_1' - \bar{\kappa} u_2)\right]' \\
&\qquad - \bar{\kappa}\left(\mathring{T} + EAe\right)(u_2' + \bar{\kappa} u_1) + \tilde{p}_1 = m\ddot{u}_1 \\
&\bar{\kappa} EAe + \left[\left(\mathring{T} + EAe\right)(u_2' + \bar{\kappa} u_1)\right]' \\
&\qquad + \bar{\kappa}\left(\mathring{T} + EAe\right)(u_1' - \bar{\kappa} u_2) + \tilde{p}_2 = m\ddot{u}_2 \\
&\left[\left(\mathring{T} + EAe\right) u_3'\right]' + \tilde{p}_3 = m\ddot{u}_3 \\
&GJ_1\left(\theta' + \bar{\kappa} u_3'\right)' - \bar{\kappa} EJ_2\left(-u_3'' + \bar{\kappa}\theta\right) + \tilde{c} = I\ddot{\theta}
\end{aligned} \qquad [6.54]$$

Concerning boundary conditions, since the field equations *lowered in order*, we cannot satisfy all of them, consistent with the fact that we ignored the boundary layers. Thus, if the cable is clamped at one end, we cannot enforce rotations θ_2 and θ_3 there; in

5. Typical values are: $E/\mathring{\sigma} = 10^3$ and $\rho_3^2/l^2 = 10^{-6}$ or smaller.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

contrast, we have to prescribe the twist. Therefore, the geometric boundary conditions [6.12] reduce to:

$$u_{1H} = \breve{u}_{1H}, \qquad u_{2H} = \breve{u}_{2H}, \qquad u_{3H} = \breve{u}_{3H}, \qquad \theta_{1H} = \breve{\theta}_{1H} \tag{6.55}$$

Analogously, at the free end of the cable, we cannot prescribe the value of the bending moment, but we have to enforce the torsional moment. Thus, the mechanical boundary conditions [6.34] reduce to:

$$\begin{aligned}
&\mp \left[EAe + \left(\mathring{T} + EAe\right)(u_1' - \bar{\kappa} u_2)\right]_H = \tilde{P}_{1H} \\
&\mp \left[\left(\mathring{T} + EAe\right)(u_2' + \bar{\kappa} u_1)\right]_H = \tilde{P}_{2H} \\
&\mp \left[\left(\mathring{T} + EAe\right) u_3'\right]_H = \tilde{P}_{3H} \\
&\mp \left[GJ_1 \left(\theta' + \bar{\kappa} u_3'\right)\right]_H = \tilde{C}_H
\end{aligned} \tag{6.56}$$

REMARK 6.4. The torsion equation [6.54d] is uncoupled from the others, so that, in most of the problems, the twist angle behaves as a passive variable, forced by the out-of-plane motion $u_3(s,t)$, which evolves according to the flexible model. However, an important exception to this rule occurs, as discussed at the beginning, namely when the incremental forces depend on the configuration, i.e. $\tilde{p}_i = \tilde{p}_i(u_1, u_2, u_3, \theta)$. In this case, the twist actively participates in the motion, since it affects the forces $\tilde{p}_i$, which drive the out-of-plane response u_3, which in turn modifies θ, so that the problem is fully coupled.

6.4.2 Shallow cables

When the sag-to-length ratio $\delta := d/l$ is small, for example of the order $\mathrm{O}\left(10^{-1}\right)$, we say that the cable is shallow, and assume that the hypotheses introduced in section 5.4 are valid, in particular $\bar{\kappa} l = \text{const} = \mathrm{O}\left(\delta^2\right)$, and $\mathring{T} = \text{const}$ along s. These allow us to further simplify the linear momentum equations, as we did for the flexible model, also performing the static condensation of the tangential displacement. Thus, when boundary conditions prescribe zero displacements, we reobtain equations [5.111] and [5.110], with symbols updated, namely:

$$\begin{aligned}
&\left(\mathring{T} + EAe_0\right) u_2'' + EAe_0 \bar{\kappa} + \tilde{p}_2 = m\ddot{u}_2 \\
&\left(\mathring{T} + EAe_0\right) u_3'' + \tilde{p}_3 = m\ddot{u}_3
\end{aligned} \tag{6.57}$$

where:

$$e_0(t) = -\frac{\bar{\kappa}}{l} \int_{s_A}^{s_B} u_2 ds + \frac{1}{2l} \int_{s_A}^{s_B} \left(u_2'^2 + u_3'^2\right) ds \tag{6.58}$$

and where $u_{2H} = u_{3H} = 0$ at $H = A, B$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Concerning instead the torsion equation [6.54d], it cannot be further simplified. Indeed, a simple analysis of equation [6.51] reveals that, although $\bar{\kappa}$ is small, $\bar{\kappa}\theta$ could be of the same order of θ'', since the twist could contain a large, almost constant component (pendulum motion), if allowed by the end constraints (e.g. spherical hinges)[6]. By taking $\bar{\kappa}$ constant, the equation becomes:

$$GJ_1\theta'' - \bar{\kappa}^2 EJ_2\theta + (GJ_1 + EJ_2)\,\bar{\kappa}u_3'' + \tilde{c} = I\ddot{\theta} \qquad [6.59]$$

A static condensation of the twist can be performed, in the same manner as we did for the tangential displacement, if we observe that the celerity of the torsional waves is much higher than that of the transverse waves[7]. Thus, if we neglect the twist inertia in the previous equation and assume $\tilde{c} = 0$ for simplicity, by solving for θ as a function of u_3, we obtain:

$$\theta = -\frac{GJ_1 + EJ_2}{\sqrt{GJ_1\,EJ_2}} \int_0^s u_3''(\sigma, t) \sinh\left[\kappa^\star (s-\sigma)\right] d\sigma + A\cosh\kappa^\star s + B\sinh\kappa^\star s \qquad [6.60]$$

where $\kappa^\star := \bar{\kappa}\sqrt{EJ_2/GJ_1}$ and A, B are arbitrary constants. These are determined by the boundary conditions for the twist. For example, if the twist is restrained, then $\theta_H = 0$; if the twist is free (spherical hinge), then $[\theta' + \bar{\kappa}u_3']_H = 0$.

6.5 Inextensible stiff cables

We now want to formulate a model of stiff cable that is able to bend and twist, but not to elongate. Consistently with the approximations of this chapter, we use linear kinematics to evaluate the changes of curvature, but exact kinematics to express the inextensibility condition.

---

6. The solution to equation [6.51] is:

$$\theta = A\cosh\bar{\kappa}s + B\sinh\bar{\kappa}s + \bar{\kappa}f(s,t)$$
$$\simeq A + \bar{\kappa}\left(Bs + f(s,t)\right) + \mathrm{O}\left(\delta^2\right)$$

where A, B are arbitrary constants, $f(s,t)$ is a particular solution to the know-term u_3'' and where we used the fact that $\bar{\kappa}s$ is small in the $[0, l]$ interval. Therefore, quasi-constant rotations $\theta \simeq A$ are in principle possible, which, however, do not affect the derivatives.

7. See [LUO 07] for a numerical example.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

6.5.1 Unprestressed cables

Inextensibility requires that the unit extension [6.7] identically vanishes at any time, so that:

$$e = \sqrt{\left(1 + u_1' - \bar{\kappa} u_2\right)^2 + \left(u_2' + \bar{\kappa} u_1\right)^2 + u_3'^2} - 1 = 0 \qquad [6.61]$$

From this, by equating $\dot{e}$ to zero, we obtain the following velocity constraint:

$$\left(1 + u_1' - \bar{\kappa} u_2\right)\left(\dot{u}_1' - \bar{\kappa} \dot{u}_2\right) + \left(u_2' + \bar{\kappa} u_1\right)\left(\dot{u}_2' + \bar{\kappa} \dot{u}_1\right) + u_3' \dot{u}_3' = 0 \qquad [6.62]$$

The VPP [6.19] still holds, in which, however, the tension T is a Lagrange multiplier associated with the inextensibility condition, namely:

$$\begin{aligned} &\int_{\mathcal{S}} \left(M_1 \dot{\chi}_1 + M_2 \dot{\chi}_2 + M_3 \dot{\chi}_3\right) ds = \left(\sum_{i=1}^{3} \bar{p}_i \dot{u}_i + \bar{c} \dot{\theta}\right) ds \\ &\quad + \sum_{H=A}^{B} \left[\sum_{i=1}^{3} \bar{P}_i \dot{u}_i + \bar{C} \dot{\theta}\right]_H - \int_{\mathcal{S}} T \dot{e} ds \qquad \forall \left(\dot{u}_1, \dot{u}_2, \dot{u}_3, \dot{\theta}\right) \end{aligned} \qquad [6.63]$$

Therefore, the same balance equations [6.22] are derived, but with $e = 0$, i.e.:

$$\begin{aligned} &\bar{\kappa} M_3' + \left[T\left(1 + u_1' - \bar{\kappa} u_2\right)\right]' - \bar{\kappa} T\left(u_2' + \bar{\kappa} u_1\right) + \bar{p}_1 = m \ddot{u}_1 \\ &-M_3'' + \left[T\left(u_2' + \bar{\kappa} u_1\right)\right]' + \bar{\kappa} T\left(1 + u_1' - \bar{\kappa} u_2\right) + \bar{p}_2 = m \ddot{u}_2 \\ &M_2'' + \left(\bar{\kappa} M_1\right)' + \left(T u_3'\right)' + \bar{p}_3 = m \ddot{u}_3 \\ &M_1' - \bar{\kappa} M_2 + \bar{c} = I \ddot{\theta} \end{aligned} \qquad [6.64]$$

Similarly, the alternative boundary conditions [6.23] are obtained:

$$\begin{aligned} &\left[\bar{P}_1 \pm \left(\bar{\kappa} M_3 + T\left(1 + u_1' - \bar{\kappa} u_2\right)\right)\right]_H \dot{u}_{1H} = 0 \\ &\left[\bar{P}_2 \pm \left(-M_3' + T\left(u_2' + \bar{\kappa} u_1\right)\right)\right]_H \dot{u}_{2H} = 0 \\ &\left[\pm M_3\right]_H \dot{u}_{2H}' = 0 \\ &\left[\bar{P}_3 \pm \left(M_2' + \bar{\kappa} M_1 + T u_3'\right)\right]_H \dot{u}_{3H} = 0 \\ &\left[\mp M_2\right]_H \dot{u}_{3H}' = 0 \\ &\left[\bar{C} \pm M_1\right]_H \dot{\theta}_H = 0 \end{aligned} \qquad [6.65]$$

In these equations, the active stresses must be expressed in terms of displacements via the elastic law [6.24b–d] and the curvature–displacement relationships [6.14], thus

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

obtaining:

$$\begin{aligned} M_1 &= GJ_1\left(\theta' + \bar{\kappa}u_3'\right) \\ M_2 &= EJ_2\left(-u_3'' + \bar{\kappa}\theta\right) \\ M_3 &= EJ_3\left(u_2'' + (\bar{\kappa}u_1)'\right) \end{aligned} \qquad [6.66]$$

The Fundamental Problem is completed by the geometric constraint [6.61] and the geometric boundary conditions [6.12]. It turns out to be a mixed problem, where the unknowns are the master configuration variables u_i, θ and the reactive tension T.

REMARK 6.5. For the flexible cable model, where no shear forces exist, we followed the mixed formulation (section 5.5); for the stiff cable model, for which shear forces instead exist, we used the hybrid formulation. As a matter of fact, we directly accounted for unshearability, via master and slave variables, but enforced inextensibility via a Lagrange multiplier. As a result, the (non-zero) reactive shear forces were removed from the balance equations, while the reactive tension, in contrast, appeared.

6.5.2 Prestressed cables

If the inextensible cable is preloaded by forces $\mathring{\mathbf{p}}, \mathring{\mathbf{c}}, \mathring{\mathbf{P}}_H, \mathring{\mathbf{C}}_H$, causing the prestresses $\mathring{M}_i, \mathring{T}$ and, moreover, it assumes a planar configuration, then we take this as a reference configuration.

The nonlinear model

Since the balance equations [6.64] and [6.65] are, by hypothesis, satisfied identically with zero displacements, the balance equations [6.27] and [6.28] must hold. By subtracting them from equations [6.64] and [6.65], the incremental balance equations [6.29] and [6.30] are obtained, with no further approximation required, since the unit extension is now rigorously zero.

The constitutive law concerns the active stresses M_i. Since they only appear as incremental parts $\tilde{M}_i$, and these are proportional to the change of curvatures [6.14], then:

$$\begin{aligned} \tilde{M}_1 &= GJ_1\left(\theta' + \bar{\kappa}u_3'\right) \\ \tilde{M}_2 &= EJ_2\left(-u_3'' + \bar{\kappa}\theta\right) \\ \tilde{M}_3 &= EJ_3\left(u_2'' + (\bar{\kappa}u_1)'\right) \end{aligned} \qquad [6.67]$$

In contrast, $T = \mathring{T} + \tilde{T}$ is a reactive stress, $\mathring{T}$ being known and $\tilde{T}$ unknown.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

When these relationships are substituted in the incremental balance equations, the following equations of motion are derived:

$$\begin{aligned}&\bar{\kappa}GJ_1\left(\theta'+\bar{\kappa}u_3'\right)'+\tilde{T}'+\left[T\left(u_1'-\bar{\kappa}u_2\right)\right]'\\&\qquad-\bar{\kappa}T\left(u_2'+\bar{\kappa}u_1\right)+\tilde{p}_1=m\ddot{u}_1\\&-EJ_3\left(u_2''+\left(\bar{\kappa}u_1\right)'\right)''+\bar{\kappa}\tilde{T}+\left[T\left(u_2'+\bar{\kappa}u_1\right)\right]'\\&\qquad+\bar{\kappa}T\left(u_1'-\bar{\kappa}u_2\right)+\tilde{p}_2=m\ddot{u}_2\\&EJ_2\left(-u_3''+\bar{\kappa}\theta\right)''+\left[\bar{\kappa}GJ_1\left(\theta'+\bar{\kappa}u_3'\right)\right]'+\left(Tu_3'\right)'+\tilde{p}_3=m\ddot{u}_3\\&GJ_1\left(\theta'+\bar{\kappa}u_3'\right)'-\bar{\kappa}EJ_2\left(-u_3''+\bar{\kappa}\theta\right)+\tilde{c}=I\ddot{\theta}\end{aligned}\qquad[6.68]$$

with the boundary conditions:

$$\begin{aligned}&\mp\left[\bar{\kappa}EJ_3\left(u_2''+\left(\bar{\kappa}u_1\right)'\right)+\tilde{T}+T\left(u_1'-\bar{\kappa}u_2\right)\right]_H=\tilde{P}_{1H}\\&\mp\left[-EJ_3\left(u_2''+\left(\bar{\kappa}u_1\right)'\right)'+T\left(u_2'+\bar{\kappa}u_1\right)\right]_H=\tilde{P}_{2H}\\&\mp\left[EJ_3\left(u_2''+\left(\bar{\kappa}u_1\right)'\right)\right]_H=0\\&\mp\left[EJ_2\left(-u_3''+\bar{\kappa}\theta\right)'+\bar{\kappa}GJ_1\left(\theta'+\bar{\kappa}u_3'\right)+Tu_3'\right]_H=\tilde{P}_{3H}\\&\pm\left[EJ_2\left(-u_3''+\bar{\kappa}\theta\right)\right]_H=0\\&\mp\left[GJ_1\left(\theta'+\bar{\kappa}u_3'\right)\right]_H=\tilde{C}_H\end{aligned}\qquad[6.69]$$

They must be supplemented with the constraint condition [6.61] and the geometric boundary conditions [6.12].

The linearized model

According to the linearized theory, the previous equations of motion are simplified by substituting T by $\mathring{T}$, namely:

$$\begin{aligned}&\bar{\kappa}GJ_1\left(\theta'+\bar{\kappa}u_3'\right)'+\tilde{T}'+\left[\mathring{T}\left(u_1'-\bar{\kappa}u_2\right)\right]'\\&\qquad-\bar{\kappa}\mathring{T}\left(u_2'+\bar{\kappa}u_1\right)+\tilde{p}_1=m\ddot{u}_1\\&-EJ_3\left(u_2''+\left(\bar{\kappa}u_1\right)'\right)''+\bar{\kappa}\tilde{T}+\left[\mathring{T}\left(u_2'+\bar{\kappa}u_1\right)\right]'\\&\qquad+\bar{\kappa}\mathring{T}\left(u_1'-\bar{\kappa}u_2\right)+\tilde{p}_2=m\ddot{u}_2\\&EJ_2\left(-u_3''+\bar{\kappa}\theta\right)''+\left[\bar{\kappa}GJ_1\left(\theta'+\bar{\kappa}u_3'\right)\right]'+\left(\mathring{T}u_3'\right)'+\tilde{p}_3=m\ddot{u}_3\\&GJ_1\left(\theta'+\bar{\kappa}u_3'\right)'-\bar{\kappa}EJ_2\left(-u_3''+\bar{\kappa}\theta\right)+\tilde{c}=I\ddot{\theta}\end{aligned}\qquad[6.70]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and:

$$\begin{aligned}
&\mp\left[\bar{\kappa}EJ_3\left(u_2''+(\bar{\kappa}u_1)'\right)+\tilde{T}+\mathring{T}\left(u_1'-\bar{\kappa}u_2\right)\right]_H=\tilde{P}_{1H}\\
&\mp\left[-EJ_3\left(u_2''+(\bar{\kappa}u_1)'\right)'+\mathring{T}\left(u_2'+\bar{\kappa}u_1\right)\right]_H=\tilde{P}_{2H}\\
&\mp\left[EJ_3\left(u_2''+(\bar{\kappa}u_1)'\right)\right]_H=0\\
&\mp\left[EJ_2\left(-u_3''+\bar{\kappa}\theta\right)'+\bar{\kappa}GJ_1\left(\theta'+\bar{\kappa}u_3'\right)+\mathring{T}u_3'\right]_H=\tilde{P}_{3H}\\
&\pm\left[EJ_2\left(-u_3''+\bar{\kappa}\theta\right)\right]_H=0\\
&\mp\left[GJ_1\left(\theta'+\bar{\kappa}u_3'\right)\right]_H=\tilde{C}_H
\end{aligned} \qquad [6.71]$$

Consistently, the geometrical constraint [6.61] is linearized too:

$$u_1'-\bar{\kappa}u_2=0 \qquad [6.72]$$

The geometric boundary conditions [6.12] are, instead, already in the linear form.

6.5.3 Reduced model

If boundary layers are not of interest, we can reduce the model by neglecting bending and torsion moments. The balance equations [6.68] reduce to:

$$\begin{aligned}
&\tilde{T}'+\left[T\left(u_1'-\bar{\kappa}u_2\right)\right]'-\bar{\kappa}T\left(u_2'+\bar{\kappa}u_1\right)+\tilde{p}_1=m\ddot{u}_1\\
&\bar{\kappa}\tilde{T}+\left[T\left(u_2'+\bar{\kappa}u_1\right)\right]'+\bar{\kappa}T\left(u_1'-\bar{\kappa}u_2\right)+\tilde{p}_2=m\ddot{u}_2\\
&\left(Tu_3'\right)'+\tilde{p}_3=m\ddot{u}_3\\
&GJ_1\left(\theta'+\bar{\kappa}u_3'\right)'-\bar{\kappa}EJ_2\left(-u_3''+\bar{\kappa}\theta\right)+\tilde{c}=I\ddot{\theta}
\end{aligned} \qquad [6.73]$$

to be joined to equation ([6.61]). The consistent geometric boundary conditions become:

$$u_{1H}=\breve{u}_{1H},\qquad u_{2H}=\breve{u}_{2H},\qquad u_{3H}=\breve{u}_{3H},\qquad \theta_{1H}=\breve{\theta}_{1H} \qquad [6.74]$$

with the mechanical boundary conditions changing into:

$$\begin{aligned}
&\mp\left[\tilde{T}+T\left(u_1'-\bar{\kappa}u_2\right)\right]_H=\tilde{P}_{1H}\\
&\mp\left[T\left(u_2'+\bar{\kappa}u_1\right)\right]_H=\tilde{P}_{2H}\\
&\mp\left[Tu_3'\right]_H=\tilde{P}_3\\
&\mp\left[GJ_1\left(\theta'+\bar{\kappa}u_3'\right)\right]_H=\tilde{C}_H
\end{aligned} \qquad [6.75]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

6.6 Summary

In this chapter, we formulated models of *stiff cables*, i.e. extremely flexible bodies, endowed with flexural and torsional stiffnesses. The need for including these effects, which were neglected in the (purely) flexible model of the cable, was primarily discussed. We identified three classes of problem in which the cable has to be considered stiff. (a) *Large amplitude oscillations*. When a cable, prestressed, e.g., by its own weight, undergoes large oscillations, the increment of stress can overcome (in absolute value) the prestress, so that the cable, in principle, experiences compression. Since the critical load of cables is almost close to zero, due to their evanescent flexural stiffness, it is expected that energy migrates from extensional to flexural form. Therefore, flexural stiffness cannot be ignored. On the other hand, due to the equilibrium of the curved element, if a bending moment arises, a torsional moment also has to be triggered, so that all the internal couples must be accounted for. Therefore, the need to use a *polar continuum* was recognized. (b) *Boundary layers*. We observed that the equations for the stiff cable are *singular equations* since a small coefficient affecting the highest derivative appears. Far from the end (or load singularities in the field), the highest derivatives can be neglected, and the flexible model can be used; however, close to the boundaries, in order to fit the boundary conditions (in larger number with respect to the flexible model), the response of the cable is strongly variable, so that the highest derivatives cannot be neglected. Here, the cable behaves as a beam. In summary, two types of regime can be observed: the *outer*, valid in the most part of the field, and the *inner*, close to the boundaries. (c) *Twist-dependent forces*. Since, in aerodynamics, the forces depend on the attitude of the body, twist cannot be ignored, although it would be of minor importance in describing the mechanics of the body itself. Therefore, again, a polar model must be adopted.

We conjectured that, in these classes of problem, a linear approximation of the curvature is sufficient to capture the phenomenon (except for the appearance of loops, which call for large curvatures). Consequently, we developed an approximated model of small-curvature large-extension cable, i.e. we used linear kinematics to express the bending and torsion and nonlinear kinematics to express the extension. Shear-strains, instead, were assumed to be rigorously zero (unshearable beam). Of course, if an exact model were instead desired, the curved beam model of Chapter 3 should be used. To limit the algebra, we assumed that the (unique) natural configuration of the stiff cable is planar. By following the displacement formulation for internally constrained beams, we eliminated two slave rotations, by linking them to the master variables (three translation components and the twist) via the unshearability conditions. Then, by using the VPP, we obtained four balance equations. Assuming no prestress, four equations of motion in the master displacements were derived, together with natural boundary conditions.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

When the cable is prestressed, under the hypothesis that the reference (prestressed) configuration is still planar, we obtained incremental balance equations as the difference between the current balance equations and the static version holding in the prestressed state. As we did for the flexible model, we neglected the unit strain with respect to 1, in order to get simpler expressions. Linearized equations, governing the small motions around the prestressed state, were also given. Furthermore, all the equations were specialized to the case of taut stiff string, both in nonlinear and linearized regimes.

An order of magnitude analysis was carried out on the equations of a prestressed cable in order to discuss the role of the additional (polar) terms appearing in the stiff *versus* the flexible model. We found that the flexural–torsional effects are negligible in the greater part of the field, so that the flexible model could be used. However, this is not true close to the boundaries, or close to load singularities, where the complete equations must be used. We noted that if we are not interested in the boundary layers, we can use a reduced model, made up of the three equations of motions of the flexible model, and an additional equation governing the twist. The latter expresses the balance of out-of-plane flexural couples, torsional couples, external incremental couples and the inertia effects. The torsion equation is uncoupled from the other three. Therefore: (a) if the external forces are independent of the twist, this is a *passive variable*, in the sense that it does not affect the motion, but it can be computed after the translational response has been evaluated by the flexible model; (b) if the external forces do depend on the twist, as happens when they are of aerodynamic nature, then the twist is an *active variable*, since it contributes to the determination of the action, which drives the out-of-plane motion, which in turn affects the twist. Of course, the reduced model can be further simplified for shallow cables, along the lines we already illustrated for the flexible model. We also noted that since the celerity of the (prevalently) torsional waves is much higher than that of the transverse waves, which we are mostly interest in, we can statically condense the twist. Accordingly, we neglected the torsional inertia, and expressed the twist as an integral of the out-of-plane motion. The procedure is similar to that which allows us to express a tangential motion of shallow flexible cables as a slave of the transverse motion. In all cases, when a reduced model is used, since the order of the differential problem is lowered with respect to the original one, not all the boundary conditions can be satisfied, consistent with the fact we ignored the boundary layers.

At the end of the chapter, we briefly addressed inextensible stiff cables. By following a hybrid formulation, in which shear forces are condensed while the tension is accounted for as a Lagrange multiplier, we obtained four equations of motion in the four master displacements and the tension, to which the inextensibility condition must be appended. If the cable is prestressed, incremental balance equations were found, in the four displacements and in the increment of the reactive tension. A reduced model was also derived by neglecting the polar effects in the *linear* momentum equations only.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Chapter 7

Locally-Deformable Thin-Walled Beams

In this chapter, we formulate a few one-dimensional (1D)-models of thin-walled beam (TWB), whose cross-section is free to deform itself, and therefore called "locally deformable beams". In the first part of the chapter, we develop a "direct" 1D-model, by introducing a discrete number of new "distortional variables", able to account for changes of the shape in the cross-sections, both in time and along the beam axis. A "two-axis beam" is also introduced, by distinguishing the centroid- and the flexural-center axes, since this shrewdness entails some simplifications in the equations. In the second part of the chapter, an identification procedure is illustrated based on a 3D fiber-model, called the "bundle of rods", in which each fiber obeys the kinematic laws established for compact beams. The analysis is not only aimed at stating a nonlinear constitutive law for the 1D-model, via an identification procedure, but also at giving a physical meaning to all the kinematic and dynamic quantities involved. The method is illustrated by referring to: (a) warpable, cross-undeformable beams, embedded in a 3D-space; and (b) unwarpable, cross-deformable beams, embedded in a 2D-space.

7.1 Motivations

So far, we considered beams (or stiff cables) whose cross-sections are planar rigid bodies. The hypothesis allowed us to reduce the original 3D problem to a simpler 1D problem, for which a direct model can be easily formulated. We referred to these as *locally rigid beams*, thereby indicating the property of the cross-sections. However, problems exist in which such an idealization is inadequate to describe the

Mathematical Models of Beams and Cables Angelo Luongo and Daniele Zulli
© 2013 ISTE Ltd. Published 2013 by ISTE Ltd.

phenomenon, and therefore call for formulating beams whose cross-sections are able to change their initial shape. We will refer to this kind of more sophisticated model as *locally deformable beams*, still alluding to the capability of the cross-sections.

As a first example, let us consider the torsion of a bar, produced by two opposite torque-couples applied at the ends. We know, from the de Saint-Venant theory, that, even in the linear range, the cross-sections warp themselves, but all in the same way, so that no longitudinal strains (and stresses) arise. This kind of deformation is called *uniform torsion*. Here, warping plays the role of a secondary effect, of the same type of the transverse deformation due to Poisson effect, occurring in flexure. In contrast, if one or both the end-sections are warping-restrained, and/or distributed twist-couples act along the beam axis, the torsion becomes *non-uniform*, so that the warping changes along the beam axis. This entails the occurrence of longitudinal stresses, which in turn call for equilibrating tangential stresses (called "secondary"), which add themselves to the "primary" ones, thus deeply modifying the mechanical behavior of the beam. The phenomenon is particularly significant in TWB, which is very flexible to torsion, and therefore sensitive to changes in the state of tangential stress. The example clearly shows how the distortion (in this case, out-of-plane) of the cross-section can play a key role, requiring a more accurate model than the locally rigid beam.

As a second example, we consider a thin tubular beam, under planar flexure. It is well-known, from experimental observations, that the annular section of the beam modifies its initial shape in an *ovalized* pattern, flattening on the plane orthogonal to the deflection plane. This phenomenon, known in literature (see [BRA 27]) as *Brazier effect* (or "ovalization" or "flattening phenomenon"), causes a magnification of the deflection of the beam, due to the reduction of the moment of inertia of the deformed cross section with respect to the neutral axis, and possibly leads to the collapse of the structure for instability (limit point bifurcation). Brazier explained the observed mechanical behavior with very simple and effective reasoning. When the beam is bent, the stress regime is governed by the well-known Navier formula, predicting a linear distribution of the normal stress on the cross-section. However, this result concerns the de Saint-Venant *linear* theory, for which equilibrium is enforced in the reference (undeformed) configuration. Since, in contrast, the beam is bent, the Navier stresses are not in equilibrium, but call for external pressures, direct along the principal normal to the bent fibers, to be (fictitiously) applied, proportional to the curvature as well as to the normal stress, and direct along the outward (inward) normal to the fiber, if they are taut (compressed), namely from the inside of the tube to the outside. Since these pressures do not really exist, we have to remove them, by applying pressures changed in sign, from the outside to inside, thus causing the flattening of the tube. The example shows how important it is for tubular beams to use a proper modeling which accounts for local (in this case, in-plane) deformability of the cross-sections.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

7.2 A one-dimensional direct model for double-symmetric TWB

We first address the task of formulating a 1D-model of TWB via a *direct approach*. To make the treatment easier, we refer to double-symmetric cross-sections, for which the centroid and the flexural center coincide, so that we can focus our attention on the "true" novelty of the model, namely the local distortion of the beam. Later, in the next section, we will study how to tackle general cross-sections.

Kinematics

Let us consider a beam, whose configuration is described by a translation field $\mathbf{u}(s,t) := \mathbf{x} - \bar{\mathbf{x}}$, a rotation field $\mathbf{R}(s,t)$ and a set of n scalar fields $(a_1(s,t), a_2(s,t), \ldots, a_n(s,t))$, each measuring an independent type of distortion of the cross-section. We will refer to them as *distortional variables*. By following the reasoning we developed about kinematics of the metamodel (section 1.2.1), there exist $6 + n$ scalar configuration variables, which, together with their first space-derivatives, constitute a set of $12 + 2n$ d.o.f for the element of beam. Since six of them describe a rigid motion, the beam can deform itself in $6 + 2n$ essentially independent ways. Therefore, n additional distortional variables a_j increase of $2n$ the number of strain components. In other words, since the rigid motions have already been accounted for in defining the strains of the locally rigid model, any additional generalized displacement a_j entails that *the same* a_j *and its first derivative* a_j' *are generalized strains*. Thus, a_j *is simultaneously a configuration variable and a strain*, since, when it is different from zero, it necessarily implies a non-rigid transformation of the beam configuration. In conclusion, the following strain-displacement-relationships hold:

$$\begin{aligned}
&\mathbf{e} := \mathbf{R}^T(\bar{\mathbf{a}}_1 + \mathbf{u}') - \bar{\mathbf{a}}_1 \\
&\mathbf{k} := \text{axial}\left[\mathbf{R}^T\mathbf{R}'\right] \\
&\alpha_j := a_j, \qquad \beta_j := a_j'
\end{aligned} \qquad [7.1]$$

The first two of them are the familiar strains of the locally rigid beam, and the last two concern the strains of the cross-section[1], namely the *distortional strain* α_j and the *distortion gradient* β_j. It should be noted that *kinematics of the rigid and deformable cross-section is uncoupled*, since $\mathbf{e}$ and $\mathbf{k}$ only depend on $\mathbf{R}$ and $\mathbf{u}$, while α_j and β_j only depend on a_j.

1. In order to stress the double nature of a_j, we introduced a different name for the strain, α_j, although it coincides with a_j.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The geometric boundary conditions prescribe the values of the configuration variables at the ends, if a constraint is applied, namely[2]:

$$\mathbf{u}_H = \breve{\mathbf{u}}_H(t), \qquad \mathbf{R}_H = \breve{\mathbf{R}}_H(t), \qquad a_{jH} = \breve{a}_{jH}, \qquad H = A, B \qquad [7.2]$$

The velocity consists of a translational velocity vector field $\mathbf{v} = \dot{\mathbf{u}}(s,t)$, an angular velocity vector field $\boldsymbol{\omega} = \text{axial}\left[\dot{\mathbf{R}}(s,t)\,\mathbf{R}^T(s,t)\right]$ and a set of scalar velocity fields $\dot{a}_j(s,t)$. By time-differentiating the previous equations, we get the strain rates $\dot{\mathbf{e}}, \dot{\mathbf{k}}, \dot{\alpha}_j, \dot{\beta}_j$. They are related to the stretching velocity gradients by:

$$\mathbf{R}\dot{\mathbf{e}} = \mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}', \qquad \mathbf{R}\dot{\mathbf{k}} = \boldsymbol{\omega}', \qquad \dot{\alpha}_j = \dot{a}_j, \qquad \dot{\beta}_j = \dot{a}_j' \qquad [7.3]$$

the first two being well-known from equations [2.79] and [2.83], and the last two being an obvious consequence of equations [7.1c,d].

Dynamics

In this section, we first consider static forces, and later account for inertia forces via the d'Alembert Principle. We consider the beam loaded by *generalized external forces*, defined as "dynamic quantities spending virtual power on the independent velocity fields", via:

$$\begin{aligned}\mathcal{P}_{ext} :=& \int_{\mathcal{S}} \left(\mathbf{p}\cdot\mathbf{v} + \mathbf{c}\cdot\boldsymbol{\omega} + \sum_{j=1}^{n} q_j \dot{a}_j\right) ds \\ &+ \sum_{H=A}^{B} \left(\mathbf{P}_H\cdot\mathbf{v}_H + \mathbf{C}_H\cdot\boldsymbol{\omega}_H + Q_{jH}\dot{a}_{jH}\right)\end{aligned} \qquad [7.4]$$

Here, $\mathbf{p}, \mathbf{P}_H$ and $\mathbf{c}, \mathbf{C}_H$ are the usual forces and couples we introduced in the locally rigid model, while q_j, Q_j are new entities, peculiar to the locally deformable beam, to be referred to as *distortional forces*. The two sets of forces should be regarded as Lagrangian external forces, respectively associated with rigid and non-rigid motions of the cross-section (equivalent to forces distributed on the cross-section of the 3D-body).

Then, by similar arguments, but referring to the internal virtual power, we introduce *generalized internal forces*, or stresses, defined as follows:

$$\mathcal{P}_{int} := \int_{\mathcal{S}} \left(\mathbf{t}\cdot\mathbf{R}\dot{\mathbf{e}} + \mathbf{m}\cdot\mathbf{R}\dot{\mathbf{k}} + \sum_{j=1}^{n}\left(D_j\dot{\alpha}_j + B_j\dot{\beta}_j\right)\right) ds \qquad [7.5]$$

2. For example, if an infinitely rigid diaphragm is present at one end, able to prevent in-plane and out-of-plane distortions of the cross-section, then $a_{jH} = 0\ \forall j$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where $\mathbf{t}, \mathbf{m}$ are the force-stress and the couple-stress already introduced, while D_j, B_j are new internal contact actions, which will be called *distortional and bi-distortional stresses*, respectively, dual of the distortional strain-rates and their spatial gradients, respectively. Once again, the two sets of stresses should be regarded as Lagrangian internal forces equivalent to the distributed stresses acting on the cross-section.

To obtain the balance equations, we equate the external and internal powers, and use the virtual power principle (VPP), by requiring that the equality is satisfied for any kinematically admissible virtual motion. By using equation [7.3], the principle reads:

$$\begin{aligned} &\int_{\mathcal{S}} \left(\mathbf{p} \cdot \mathbf{v} + \mathbf{c} \cdot \boldsymbol{\omega} + \sum_{j=1}^{n} q_j \dot{a}_j \right) ds \\ &\quad + \sum_{H=A}^{B} \left(\mathbf{P}_H \cdot \mathbf{v}_H + \mathbf{C}_H \cdot \boldsymbol{\omega}_H + Q_{jH} \dot{a}_{jH} \right) \\ &\quad = \int_{\mathcal{S}} \left(\mathbf{t} \cdot (\mathbf{v}' - \boldsymbol{\omega} \times \mathbf{x}') + \mathbf{m} \cdot \boldsymbol{\omega}' + \sum_{j=1}^{n} \left(D_j \dot{a}_j + B_j \dot{a}_j' \right) \right) ds \\ &\quad \forall \left(\mathbf{v}, \boldsymbol{\omega}, \dot{a}_j \right) \end{aligned} \qquad [7.6]$$

or, after integration by parts:

$$\begin{aligned} &\int_{\mathcal{S}} \left((\mathbf{t}' + \mathbf{p}) \cdot \mathbf{v} + (\mathbf{m}' + \mathbf{x}' \times \mathbf{t} + \mathbf{c}) \cdot \boldsymbol{\omega} + \sum_{j=1}^{n} \left(B_j' - D_j + q_j \right) \dot{a}_j \right) ds + \\ &\sum_{H=A}^{B} \left[(\mathbf{P}_H \pm \mathbf{t}_H) \cdot \mathbf{v}_H + (\mathbf{C}_H \pm \mathbf{m}_H) \cdot \boldsymbol{\omega}_H + (Q_{jH} \pm B_{jH}) \dot{a}_{jH} \right] = 0 \\ &\forall \left(\mathbf{v}, \boldsymbol{\omega}, \dot{a}_j \right) \end{aligned} \qquad [7.7]$$

From this expression, the balance equations follow:

$$\begin{aligned} &\mathbf{t}' + \mathbf{p} = \mathbf{0} \\ &\mathbf{m}' + \mathbf{x}' \times \mathbf{t} + \mathbf{c} = \mathbf{0} \\ &B_j' - D_j + q_j = 0, \quad j = 1, 2, \ldots n \end{aligned} \qquad [7.8]$$

together with the boundary conditions:

$$\begin{aligned} &(\mathbf{P}_H \pm \mathbf{t}_H) \cdot \mathbf{v}_H = \mathbf{0} \\ &(\mathbf{C}_H \pm \mathbf{m}_H) \cdot \boldsymbol{\omega}_H = \mathbf{0} \\ &(Q_{jH} \pm B_{jH}) \dot{a}_{jH} = 0 \end{aligned} \qquad [7.9]$$

which are, alternatively, of geometrical or mechanical type.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The first two of equations [7.8] and [7.9] are identical to those for the locally-rigid beam; the novelty consists of the balance equations [7.8c] and [7.9c], which involve the distortional stresses. It should be noted that *the locally rigid dynamics is uncoupled by the cross-section distortional dynamics*, since stresses $\mathbf{t}, \mathbf{m}$ only depend on forces $\mathbf{p}, \mathbf{P}_H, \mathbf{c}, \mathbf{C}_H$, while stresses D_j, B_j only depend on forces q_j, Q_{jH}.

Inertia forces

To account for inertia forces, we use the d'Alembert Principle. Accordingly, the generalized forces in the balance equations [7.8] must be expressed as an active and an inertia contributions, namely: $\mathbf{p} = \mathbf{p}^a + \mathbf{p}^{in}$, $\mathbf{c} = \mathbf{c}^a + \mathbf{c}^{in}$, $q_j = q_j^a + q_j^{in}$.

In Chapter 2 (equations [2.127] and [2.131]), we found that the inertia action is equivalent to: (a) a force $\mathbf{p}^{in} := -m\dot{\mathbf{v}}$, equal to the opposite to the time-derivative of the linear momentum (equation [2.119]); and (b) a couple $\mathbf{c}^{in} := -\mathbf{J}_G\dot{\boldsymbol{\omega}}$, equal to the opposite of the time-derivative of the rotational part of the angular moment (equation [2.120]). By keeping the analogy with the forces $\mathbf{p}$, we admit that the inertial distortional forces are proportional to the time-derivative of the distortional velocity $\dot{a}_j$, i.e. $q_j^{in} := -m_j\ddot{a}_j$, through a "distortional mass" m_j.

By accounting for the inertia forces, the balance equations [7.8] are thus modified[3]:

$$\begin{aligned}
&\mathbf{t}' + \mathbf{p} = m\dot{\mathbf{v}} \\
&\mathbf{m}' + \mathbf{x}' \times \mathbf{t} + \mathbf{c} = \mathbf{J}_G\dot{\boldsymbol{\omega}} \\
&B_j' - D_j + q_j = m_j\ddot{a}_j, \quad j = 1, 2, \ldots, n
\end{aligned} \qquad [7.10]$$

where the superscript on the active forces has been omitted.

Elastic law

We consider the beam as being made of a hyperelastic material. To formulate a constitutive law, first, we have to write an elastic potential ϕ, depending on the generalized strains, i.e.:

$$\phi = \phi\left(\mathbf{e}, \mathbf{k}, \alpha_j, \beta_j\right) \qquad [7.11]$$

3. Usually, however, the inertial effect of the distortion of the cross-section is kept small, and therefore neglected.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By equating the linear density of the deformation work:

$$\frac{d}{ds}\left(\mathcal{P}_{int}dt\right) = \left[\mathbf{t}\cdot\mathbf{R}\dot{\mathbf{e}} + \mathbf{m}\cdot\mathbf{R}\dot{\mathbf{k}} + \sum_{j=1}^{n}\left(D_j\dot{\alpha}_j + B_j\dot{\beta}_j\right)\right]dt \quad [7.12]$$

to the differential $d\phi$ of the elastic potential, the stress–strain relationships follow (compare them with equations [2.151], holding for the locally rigid beam):

$$\mathbf{R}^T\mathbf{t} = \frac{\partial\phi}{\partial\mathbf{e}},\quad \mathbf{R}^T\mathbf{m} = \frac{\partial\phi}{\partial\mathbf{k}},\quad D_j = \frac{\partial\phi}{\partial\alpha_j},\quad B_j = \frac{\partial\phi}{\partial\beta_j} \quad [7.13]$$

Such relations are, in general, nonlinear, but, more importantly, they *couple all stresses and strains*.

The Fundamental Problem

The Fundamental Problem is governed by the kinematic relationships (equations [7.1] and [7.3]), the balance equations (equations [7.10] and [7.9]) and the elastic law (equations [7.13]), plus the geometric boundary conditions.

REMARK 7.1. While kinematics and dynamics leave the quantities concerning locally rigid and locally deformable beams uncoupled, the elastic law finally couples them, so that the resulting equations are indeed coupled.

7.3 A one-dimensional direct model for non-symmetric TWB

When the cross-section of the TWB is not symmetric, the flexural center C of the cross-section[4] does not coincide with the centroid G. Of course, we could still refer to the centroid, by using the model of the previous section. However, while kinematic and dynamic relationships, first established, are unaffected by this choice, the constitutive law linking quantities referred to the centroid is more involved, since flexure and torsion couple themselves even in the linear range. To avoid this occurrence, it is suitable to choose the flexural center locus as the beam axis (to be referred to as the *flexural-axis*) instead of the customary centroid locus (the *centroid-axis*). Thus, a pure torsion, induced by twist-couples, leaves the position of the flexural-axis (no translation) unaltered, while a pure flexure, induced by transverse forces applied at the flexural-axis, does not entail twist. Furthermore, the

4. The flexural center is also known as “shear center” or “torsion center”.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

choice of the flexural center as a pole entails some drawbacks concerning extension and flexure induced by axial loads, which are better described with respect to the centroid. Therefore, according to what is done in the linear theory, we will favor the flexural-axis (since transverse forces are usually more important than axial forces), but we will also use the centroid-axis as a secondary one. In conclusion, non-symmetric TWB could be referred to as *two-axis beams*[5].

The contents of this section have partially been inspired by the works [RUT 06, PIG 09, RIZ 96, RUT 06, LOF 13], where, however, distortion is specifically referred to by warping only.

C-strains

Let us describe the current configuration of the beam via: (a) the position $\mathbf{x}_C\left(s,t\right)$ of its flexural-axis; (b) the tensor $\mathbf{R}\left(s,t\right)$, which represents a field of rotations occurring around axes crossing the flexural-axis; and (c) a set of scalar distortions $\left(a_1\left(s,t\right),a_2\left(s,t\right),\ldots,a_n\left(s,t\right)\right)$. By denoting by $\mathbf{u}_C\left(s,t\right):=\mathbf{x}_C-\bar{\mathbf{x}}_C$ the translation field, the following strain–displacement relationships hold (compare them with equation [7.1]):

$$\begin{aligned}\mathbf{e}_C &:= \mathbf{R}^T\left(\bar{\mathbf{a}}_1+\mathbf{u}_C'\right)-\bar{\mathbf{a}}_1\\ \mathbf{k} &:= \text{axial}\left[\mathbf{R}^T\mathbf{R}'\right]\\ \alpha_j &:= a_j, \qquad \beta_j := a_j'\end{aligned} \qquad [7.14]$$

Here, $\mathbf{e}_C,\mathbf{k}$ are the strains referred to by the flexural-axis or, in short, *C-strains*; they admit the following decomposition in $\bar{\mathcal{B}}$:

$$\begin{aligned}\mathbf{e}_C &:= \varepsilon_C\bar{\mathbf{a}}_1+\boldsymbol{\gamma}_C\\ \mathbf{k} &:= \kappa_1\bar{\mathbf{a}}_1+\mathbf{k}_\pi\end{aligned} \qquad [7.15]$$

where $\boldsymbol{\gamma}_C:=\gamma_{2C}\bar{\mathbf{a}}_2+\gamma_{3C}\bar{\mathbf{a}}_3$, $\mathbf{k}_\pi:=\kappa_2\bar{\mathbf{a}}_2+\kappa_3\bar{\mathbf{a}}_3$ are the components of $\mathbf{e}_C$ and $\mathbf{k}$ along the cross-section plane $\pi:=\text{span}\left(\bar{\mathbf{a}}_2,\bar{\mathbf{a}}_3\right)$ in the reference configuration. By remembering equations [2.52] and [2.53], and replacing u_i by u_{iC}, the scalar

5. These considerations, of course, are valid for any cross-section shapes, not necessary thin-walled. However, the question is negligible for compact beams, since points G and C are close, and the torsional stiffness of the beam is high.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

components of the strains read:

$$\begin{aligned}
\varepsilon_C &= (1+u'_{1C})(\cos\theta_2\cos\theta_3) + u'_{2C}\cos\theta_2\sin\theta_3 - u'_{3C}\sin\theta_2 - 1\\
\gamma_{2C} &= (1+u'_{1C})(\sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3)\\
&\quad + u'_{2C}(\sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3) + u'_{3C}\sin\theta_1\cos\theta_2\\
\gamma_{3C} &= (1+u'_{1C})(\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3)\\
&\quad + u'_{2C}(\cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3) + u'_{3C}\cos\theta_1\cos\theta_2\\
\kappa_1 &= \theta'_1 - \theta'_3\sin\theta_2\\
\kappa_2 &= \theta'_2\cos\theta_1 + \theta'_3\sin\theta_1\cos\theta_2\\
\kappa_3 &= -\theta'_2\sin\theta_1 + \theta'_3\cos\theta_1\cos\theta_2
\end{aligned} \qquad [7.16]$$

Velocity and strain-rates

The velocity consists of a translational velocity vector field $\mathbf{v}_C = \dot{\mathbf{u}}_C(s,t)$, an angular velocity vector field $\boldsymbol{\omega} = \text{axial}\left[\dot{\mathbf{R}}(s,t)\,\mathbf{R}^T(s,t)\right]$ and a set of scalar velocity fields $\dot{a}_j(s,t)$. The strain-rates are related to the stretching velocity gradients by:

$$\mathbf{R}\dot{\mathbf{e}}_C = \mathbf{v}'_C - \boldsymbol{\omega}\times\mathbf{x}'_C, \qquad \mathbf{R}\dot{\mathbf{k}} = \boldsymbol{\omega}', \qquad \dot{\alpha}_j = \dot{a}_j, \qquad \dot{\beta}_j = \dot{a}'_j \qquad [7.17]$$

according to equations [2.79], [2.83], and equations [7.14c,d].

Relation between C- and G-strains

As an alternative, we could refer the strains to the centroid-axis (*G-strains*). Thus, we should take $\mathbf{x}_G(s,t)$ as position vector, $\mathbf{u}_G(s,t) := \mathbf{x}_G - \bar{\mathbf{x}}_G$ as translation field and still $\mathbf{R}(s,t)$ and a_j. As a result, only the first of equations [7.14] would change into:

$$\mathbf{e}_G := \mathbf{R}^T(\bar{\mathbf{a}}_1 + \mathbf{u}'_G) - \bar{\mathbf{a}}_1 \qquad [7.18]$$

To find a relationship between the two strain measures, $\mathbf{e}_C, \mathbf{e}_G$, we observe that, in the rigid part of the transformation (i.e. to within the effect of the distortion), it is:

$$\mathbf{u}_G = \mathbf{u}_C - (\mathbf{R}-\mathbf{I})\,\bar{\mathbf{r}}_C \qquad [7.19]$$

where $\bar{\mathbf{r}}_C := \overrightarrow{GC} = r_{2C}\bar{\mathbf{a}}_2 + r_{3C}\bar{\mathbf{a}}_3$ is the oriented distance of C from G in the reference configuration. By space-differentiating the latter, substituting it in equation [7.18] and accounting for equations [7.14a,b], it follows that:

$$\mathbf{e}_G = \mathbf{e}_C - \mathbf{k}\times\bar{\mathbf{r}}_C \qquad [7.20]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By decomposing this relation along $\bar{\mathbf{a}}_1$ and π, and accounting for equation [7.15b], we finally get:

$$\begin{aligned} \varepsilon_G &= \varepsilon_C - \mathbf{k}_\pi \times \bar{\mathbf{r}}_C \cdot \bar{\mathbf{a}}_1 \\ \boldsymbol{\gamma}_G &= \boldsymbol{\gamma}_C - \kappa_1 \bar{\mathbf{a}}_1 \times \bar{\mathbf{r}}_C \end{aligned} \qquad [7.21]$$

Since using ε_G is more convenient than ε_C, while using $\boldsymbol{\gamma}_C$ is more convenient than $\boldsymbol{\gamma}_G$, we adopt a *mixed description*, by taking $\varepsilon_G, \boldsymbol{\gamma}_C$ as strain measures. Therefore, the strain is described by $(\varepsilon_G, \boldsymbol{\gamma}_C; \kappa_1, \mathbf{k}_\pi; \alpha_j, \beta_j)$.

External forces, stresses and balance equations

We define the quantities spending virtual power on the independent velocity fields $(\mathbf{v}_C, \boldsymbol{\omega}, \dot{a}_j)$ as *generalized external forces*; hence:

$$\begin{aligned} \mathcal{P}_{ext} := &\int_{\mathcal{S}} \left(\mathbf{p} \cdot \mathbf{v}_C + \mathbf{c} \cdot \boldsymbol{\omega} + \sum_{j=1}^{n} q_j \dot{a}_j \right) ds \\ &+ \sum_{H=A}^{B} \left(\mathbf{P}_H \cdot \mathbf{v}_{CH} + \mathbf{C}_H \cdot \boldsymbol{\omega}_H + \sum_{j=1}^{n} Q_{jH} \dot{a}_{jH} \right) \end{aligned} \qquad [7.22]$$

Accordingly, $\mathbf{p}, \mathbf{P}_H$ and $\mathbf{c}, \mathbf{C}_H$ are forces reduced to the pole C.

Similarly, we introduce *generalized internal forces*, or stresses $(N, \mathbf{t}_\pi; M_1, \mathbf{m}_\pi; \alpha_j, \beta_j)$, as dual quantities of the strain rates, defined via the internal virtual power expression[6]:

$$\begin{aligned} \mathcal{P}_{int} := &\int_{\mathcal{S}} \Bigg(N \dot{\varepsilon}_G + \mathbf{t}_\pi \cdot \mathbf{R} \dot{\boldsymbol{\gamma}}_C + M_1 \dot{\kappa}_1 + \mathbf{m}_\pi \cdot \mathbf{R} \dot{\mathbf{k}}_\pi \\ &+ \sum_{j=1}^{n} \left(D_j \dot{\alpha}_j + B_j \dot{\beta}_j \right) \Bigg) ds \end{aligned} \qquad [7.23]$$

Accordingly, *N is the normal stress reduced at the centroid (and consequently $\mathbf{m}_\pi$ is the flexural moment with respect to the same point G)*, while *$\mathbf{t}_\pi$ is the shear-stress*

6. Note that, consistently with the usual representation of the stresses and strains, while $\mathbf{k}_\pi$ belongs to the $(\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2)$-plane, $\mathbf{m}_\pi$ belongs to the $(\mathbf{a}_1, \mathbf{a}_2)$-plane.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

reduced to the flexural center (and consequently M_1 is the torsional moment with respect to the same point C).

By virtue of equation [7.21a], we can transform the previous expression into:

$$\begin{aligned}
\mathcal{P}_{int} := \int_{\mathcal{S}} \Bigg(& N\dot{\varepsilon}_C + \mathbf{t}_\pi \cdot \mathbf{R}\dot{\boldsymbol{\gamma}}_C + M_1\dot{\kappa}_1 \\
& + (\mathbf{m}_\pi - \mathbf{r}_C \times \mathbf{a}_1 N) \cdot \mathbf{R}\dot{\mathbf{k}}_\pi + \sum_{j=1}^{n} \left(D_j\dot{\alpha}_j + B_j\dot{\beta}_j \right) \Bigg) ds \\
= \int_{\mathcal{S}} \Bigg(& \mathbf{t} \cdot \mathbf{R}\dot{\mathbf{e}}_C + \mathbf{m} \cdot \mathbf{R}\dot{\mathbf{k}} - \mathbf{r}_C \times N\mathbf{a}_1 \cdot \mathbf{R}\dot{\mathbf{k}} \\
& + \sum_{j=1}^{n} \left(D_j\dot{\alpha}_j + B_j\dot{\beta}_j \right) \Bigg) ds \\
= \int_{\mathcal{S}} \Bigg(& \mathbf{t} \cdot (\mathbf{v}'_C - \boldsymbol{\omega} \times \mathbf{x}'_C) + (\mathbf{m} - \mathbf{r}_C \times N\mathbf{a}_1) \cdot \boldsymbol{\omega}' \\
& + \sum_{j=1}^{n} \left(D_j\dot{a}_j + B_j\dot{a}'_j \right) \Bigg) ds
\end{aligned} \qquad [7.24]$$

where in the first line we used a known property of the mixed product[7]; in the second line, we defined $\mathbf{t} := N\mathbf{a}_1 + \mathbf{t}_\pi,\ \mathbf{m} := M_1\mathbf{a}_1 + \mathbf{m}_\pi$ and grouped terms[8]; in the third

7. Here, we use $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \mathbf{R}\mathbf{u} \times \mathbf{R}\mathbf{v} \cdot \mathbf{R}\mathbf{w}$.

8. We grouped two pairs of terms, according to:

$$\mathbf{t} \cdot \mathbf{R}\dot{\mathbf{e}}_C = (N\mathbf{a}_1 + \mathbf{t}_\pi) \cdot \mathbf{R}\left(\dot{\varepsilon}_C\bar{\mathbf{a}}_1 + \dot{\boldsymbol{\gamma}}_C\right) = N\dot{\varepsilon}_C + \mathbf{t}_\pi \cdot \mathbf{R}\dot{\boldsymbol{\gamma}}_C$$

$$\mathbf{m} \cdot \mathbf{R}\dot{\mathbf{k}} = (M_1\mathbf{a}_1 + \mathbf{m}_\pi) \cdot \mathbf{R}\left(\dot{\kappa}_1\bar{\mathbf{a}}_1 + \dot{\mathbf{k}}_\pi\right) = M_1\dot{\kappa}_1 + \mathbf{m}_\pi \cdot \mathbf{R}\dot{\mathbf{k}}_\pi$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

line, we introduced equations [7.17]. Hence, the VPP, after integration by parts, reads:

$$\int_{\mathcal{S}} \left[(\mathbf{t}' + \mathbf{p}) \cdot \mathbf{v}_C + \left((\mathbf{m} - \mathbf{r}_C \times N\mathbf{a}_1)' + \mathbf{x}_C' \times \mathbf{t} + \mathbf{c} \right) \cdot \boldsymbol{\omega} + \sum_{j=1}^{n} \left(B_j' - D_j + q_j \right) \dot{a}_j \right] ds$$
$$+ \sum_{H=A}^{B} \left[(\mathbf{P}_H \pm \mathbf{t}_H) \cdot \mathbf{v}_{CH} + (\mathbf{C}_H \pm (\mathbf{m}_H - \mathbf{r}_C \times N_H \mathbf{a}_1)) \cdot \boldsymbol{\omega}_H + \sum_{j=1}^{n} (Q_{jH} \pm B_j) \dot{a}_{jH} \right] = 0, \qquad \forall (\mathbf{v}_C, \boldsymbol{\omega}, \dot{a}_j) \qquad [7.25]$$

from which the balance equations follow:

$$\begin{aligned} &\mathbf{t}' + \mathbf{p} = \mathbf{0} \\ &(\mathbf{m} - \mathbf{r}_C \times N\mathbf{a}_1)' + \mathbf{x}_C' \times \mathbf{t} + \mathbf{c} = \mathbf{0} \\ &B_j' - D_j + q_j = 0 \end{aligned} \qquad [7.26]$$

with the alternative boundary conditions:

$$\begin{aligned} &(\mathbf{P}_H \pm \mathbf{t}_H) \cdot \mathbf{v}_{CH} = \mathbf{0} \\ &(\mathbf{C}_H \pm (\mathbf{m}_H - \mathbf{r}_C \times N_H \mathbf{a}_1)) \cdot \boldsymbol{\omega}_H = \mathbf{0} \\ &(Q_{jH} \pm B_{jH}) \dot{a}_{jH} = 0 \end{aligned} \qquad [7.27]$$

The balance equations relevant to the distortion (equations [7.26c] and [7.27c]) are unaffected by the change of pole; the same occurs for the force balance (equations [7.26a], [7.27a]). In contrast, the moment equilibrium (equations [7.26b] and [7.27b]) contains extra-terms, due to the fact that moments are evaluated with respect to C [9]; hence, since the normal force is applied to the centroid G, it gives the (purely flexural) contribution $\overrightarrow{CG} \times N\mathbf{a}_1$ to the moment with respect to C.

When balance equations [7.26] and [7.27] are projected onto the *current* basis $\mathcal{B}$, scalar equations similar to equations [2.137] and [2.139], but with few extra-terms,

9. This is consistent with the fact that the rigid virtual motion has been described with respect to C.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

are obtained, together with uncoupled appended equations, namely:

$$\begin{aligned}
&N' - \kappa_3 T_2 + \kappa_2 T_3 + p_1 = 0\\
&T_2' + \kappa_3 N - \kappa_1 T_3 + p_2 = 0\\
&T_3' - \kappa_2 N + \kappa_1 T_2 + p_3 = 0\\
&M_1' - \kappa_3 M_2 + \kappa_2 M_3 + r_{2C}\kappa_2 N + r_{3C}\kappa_3 N - \gamma_{3C} T_2 + \gamma_{2C} T_3 + c_1 = 0\\
&M_2' + \kappa_3 M_1 - \kappa_1 M_3 - r_{2C}\kappa_1 N - r_{3C} N' + \gamma_{3C} N\\
&- (1 + \varepsilon_C)\, T_3 + c_2 = 0\\
&M_3' - \kappa_2 M_1 + \kappa_1 M_2 - r_{3C}\kappa_1 N + r_{2C} N' - \gamma_{2C} N\\
&+ (1 + \varepsilon_C)\, T_2 + c_2 = 0\\
&B_j' - D_j + q_j = 0
\end{aligned} \tag{7.28}$$

and:

$$\begin{aligned}
&\left[(P_1 \pm N)\, v_{1C}\right]_H = 0\\
&\left[(P_2 \pm T_2)\, v_{2C}\right]_H = 0\\
&\left[(P_3 \pm T_3)\, v_{3C}\right]_H = 0\\
&\left[(C_1 \pm M_1)\, \omega_1\right]_H = 0\\
&\left[(C_2 \pm (M_2 - N r_{3C}))\, \omega_2\right]_H = 0\\
&\left[(C_3 \pm (M_3 + N r_{2C}))\, \omega_3\right]_H = 0\\
&\left[(Q_j \pm B_j)\, \dot{a}_j\right]_H = 0
\end{aligned} \tag{7.29}$$

REMARK 7.2. The reader should clearly bear in mind that, as a consequence of the kinematic description, while stresses are referred to two poles, G and C, external forces are referred to the unique pole C.

Inertia forces

As we recalled in the previous section, the inertia action, according to d'Alembert, consists of a force $\mathbf{p}^{in} := -m\dot{\mathbf{v}}_G$, applied to the centroid, and a couple $\mathbf{c}^{in} := -\mathbf{J}_G\dot{\boldsymbol{\omega}}$. Since we are now expressing equilibrium of the moments around the pole C, we have to modify the inertia couple $\mathbf{c}^{in}$ by accounting for the eccentricity of $\mathbf{p}^{in}$; hence, $\mathbf{c}^{in} := \mathbf{r}_C \times m\dot{\mathbf{v}}_G - \mathbf{J}_G\dot{\boldsymbol{\omega}}$. Since, moreover, from equation [7.19], it follows that $\mathbf{v}_G = \mathbf{v}_C - \boldsymbol{\omega} \times \mathbf{r}_C$, we finally have:

$$\begin{aligned}
\mathbf{p}^{in} &:= -m\left(\mathbf{v}_C - \boldsymbol{\omega} \times \mathbf{r}_C\right)^{\cdot}\\
\mathbf{c}^{in} &:= \mathbf{r}_C \times m\left(\mathbf{v}_C - \boldsymbol{\omega} \times \mathbf{r}_C\right)^{\cdot} - \mathbf{J}_G\dot{\boldsymbol{\omega}}
\end{aligned} \tag{7.30}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The balance equations [7.26] therefore modify into:

$$\begin{aligned}
&\mathbf{t}' + \mathbf{p} = m\left(\mathbf{v}_C - \boldsymbol{\omega} \times \mathbf{r}_C\right)^{\cdot} \\
&\left(\mathbf{m} - \mathbf{r}_C \times N\mathbf{a}_1\right)' + \mathbf{x}'_C \times \mathbf{t} + \mathbf{c} = -\mathbf{r}_C \times m\left(\mathbf{v}_C - \boldsymbol{\omega} \times \mathbf{r}_C\right)^{\cdot} + \mathbf{J}_G\dot{\boldsymbol{\omega}} \quad [7.31] \\
&B'_j - D_j + q_j = 0
\end{aligned}$$

where the superscript on the active forces has been omitted.

Elastic law

The elastic law is still formally given by equations [7.13], but uncoupling is expected in the linear part, among axial force, flexural and torsional moments. The Fundamental Problem then follows.

7.4 Identification strategy from 3D-models of TWB

In formulating a 1D-model of TWB, we found that the direct approach is a powerful strategy, easy and elegant. However, it leaves open several questions which need an answer, before the model can be used for practical purposes, namely:

1) First, and most importantly, how to choose the constants appearing in the elastic potential?

2) What is the physical meaning of the distortional variables a_j, we introduced in the model?

3) What is the physical meaning of the generalized external forces q_j, Q_j which spend power on $\dot{a}_j$?

4) What is the physical meaning of the generalized stresses D_j, B_j, spending power on the distortional strain-rates $\dot{\alpha}_j, \dot{\beta}_j$?

The same issues, of course, could be raised by dealing with a standard, locally rigid, model. However, and only in this case, most of the previous questions have an immediate answer, since we are "accustomed" to handling rigid bodies. Namely, generalized displacements of a section are the six d.o.f. of a rigid body; the associated generalized external forces are the *resultant* and the *resultant moment* of the distributed forces, obtained by standard operation of static equivalence; similarly, the generalized stresses are the resultant and resultant moment of the local stresses, once they have been integrated over the cross-sections. The same constitutive law is a consequence of the local behavior, after local quantities have been expressed in terms of generalized quantities. All these answers, indeed, are still valid in handling locally deformable beams, with the only difference that the "weight functions" we have to

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

use to combine local forces and local stresses are *not* the usual linear functions of the coordinates (consequence of the local rigidity), but richer functions of the coordinates and of the strain themselves (consequence of the local flexibility).

We will address this topic by explaining how to use a suitable 3D-model to identify all the quantities appearing in the 1D-model. As a general philosophy, we will look for few maps, namely a *displacement*-, a *velocity*-, a *strain*- and a *strain-rate-map* which link local quantities from one side (namely displacement, velocities, strains and strain-rates at a generic point of the 3D-model), and generalized quantities on the other side (i.e. the same quantities for the 1D-model). They will be found to assume the following symbolic forms:

$$\begin{aligned}
&\mathbf{u} = \hat{\mathbf{u}}\left(\mathbf{u}_C, \mathbf{R}, a_j; \bar{\mathbf{r}}\right) \\
&\boldsymbol{\varepsilon} = \hat{\boldsymbol{\varepsilon}}\left(\mathbf{e}, \mathbf{k}, \alpha_j, \beta_j; \bar{\mathbf{r}}\right) \\
&\mathbf{v} = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r} + \sum_{j=1}^{n} \frac{\partial \hat{\mathbf{u}}}{\partial a_j} \dot{a}_j \\
&\dot{\boldsymbol{\varepsilon}} = \frac{\partial \hat{\boldsymbol{\varepsilon}}}{\partial \mathbf{e}} \dot{\mathbf{e}} + \frac{\partial \hat{\boldsymbol{\varepsilon}}}{\partial \mathbf{k}} \dot{\mathbf{k}} + \sum_{j=1}^{n} \left(\frac{\partial \hat{\boldsymbol{\varepsilon}}}{\partial \alpha_j} \dot{\alpha}_j + \frac{\partial \hat{\boldsymbol{\varepsilon}}}{\partial \beta_j} \dot{\beta}_j \right)
\end{aligned} \quad [7.32]$$

where (a) local displacements $\mathbf{u}$ and local strains $\boldsymbol{\varepsilon}$ are *nonlinear* functions of the generalized displacements $\mathbf{u}_C, \mathbf{R}, a_j$ and generalized strains $\mathbf{e}, \mathbf{k}, \alpha_j, \beta_j$, respectively; (b) local velocities $\mathbf{v}$ and local strain-rates $\dot{\boldsymbol{\varepsilon}}$, obtained by time-differentiation of the former relationships, are, of course, *linear* functions of the generalized velocities $\mathbf{v}_C, \boldsymbol{\omega}, \dot{a}_j$ and of the generalized strain-rates $\dot{\mathbf{e}}, \dot{\mathbf{k}}, \dot{\alpha}_j, \dot{\beta}_j$, respectively. In all maps, the position vector $\bar{\mathbf{r}}$ or $\mathbf{r}$ appears, evaluated in the reference or current configuration, respectively.

By equating the *external virtual power* for the two models, and using the velocity-map, the generalized external forces are identified. Similarly, by equating the *internal virtual power* for the two models, and using the strain-rate-map, the generalized stresses are identified. Finally, by equating the *elastic potential* for the two models, and using the strain-map, the elastic constants are identified. The first two steps are valid for any material; the third one, just for hyperelastic material. In what follows, we will show in detail how to apply the procedure.

7.5 A fiber-model of TWB

For identification purposes, we need a 3D-model of TWB. We will formulate one here, based on the fundamental idea of the Generalized Beam Theory (GBT), which

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

has recently received a strong following in the literature (see, e.g., [SIL 03, CAM 06])[10]. However, we will limit ourselves to exploiting this idea, by adapting the treatment to our purposes.

Geometry and basic hypotheses

Let us consider a TWB beam in the undeformed reference configuration. It is a cylinder, whose cross-section $\mathcal{A}$ (Figure 7.1) is spanned by a segment Ξ (the *chord*), of length $b = b(c)$ (the *thickness*, generally variable), which moves in the plane by keeping (a) its middle point on a planar curve $\mathcal{C}$ (the *middle-line*) and (b) its attitude orthogonal to $\mathcal{C}$. Here, c is a *curvilinear abscissa* (the *directrix-abscissa*) defined on the middle-line. We assume that the thickness $b(c)$ is everywhere much lesser than a characteristic length of the cross-section (e.g. the average diameter). This geometric property allows us to assume that, on the generic cross-section at the axis-abscissa s, *all the quantities* $\mathcal{Q}$ *relevant to points on the same chord* Ξ *are constant*, i.e. $\mathcal{Q}$ depend only on the abscissas s and c. Hence, the dimension of the spatial domain in which the quantities are defined reduces to 2. In other words, the body is confused with the cylindrical middle-surface, of directrix $\mathcal{C}$.

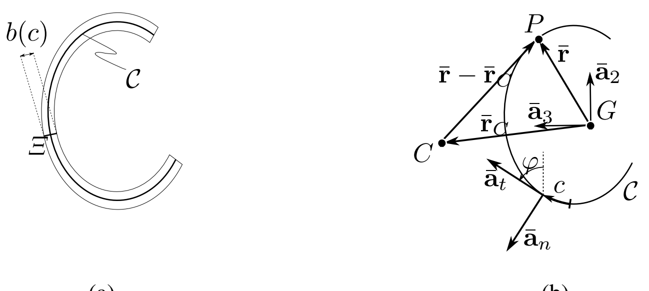


Figure 7.1: TWB cross-section: (a) geometry and chord; (b) curvilinear abscissa c, Frenet triad, centroid G and flexural center C.

A Frenet triad $(\bar{\mathbf{a}}_t(c), \bar{\mathbf{a}}_n(c), \bar{\mathbf{a}}_b)$, intrinsic to $\mathcal{C}$, is introduced in the reference configuration, in which $\bar{\mathbf{a}}_t(c)$ is the tangent unit vector, $\bar{\mathbf{a}}_n(c)$ is the normal unit vector and $\bar{\mathbf{a}}_b \equiv \bar{\mathbf{a}}_1$ is the binormal unit vector, independent of c and coincident with the normal to the cross-section.

10. GBT is a variant of the *Kantorovitch semi-variational method*, able to reduce the spatial dimension of the problem, via the partially assumed mode technique.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Distortional variables

Let us consider the TWB beam as a deformable cylinder. Its generic material point P, in the reference configuration, occupies the position $\bar{\mathbf{x}}(s,c) := s\bar{\mathbf{a}}_1 + \bar{\mathbf{r}}(c)$, where $\bar{\mathbf{r}}(c) := r_2(c)\,\bar{\mathbf{a}}_2 + r_3(c)\,\bar{\mathbf{a}}_3 \equiv \overrightarrow{GP}$ is the oriented distance of P from the centroid G of the cross-section containing P. To describe the current position $\mathbf{x}(s,\bar{\mathbf{r}}(c),t)$ of P, we can superimpose two transformations: (a) a *locally rigid transformation*, in which the cross-section behaves as a planar rigid body (as we saw, more conveniently referred to the flexural center C, instead of the centroid G); and (b) a *pure distortion*, in which the cross-section changes its shape, both in-plane and out-of-plane. This transformation is more suitably described in the reference configuration, and is then pushed forward to the current configuration. Accordingly, we have[11]:

$$\mathbf{x} = \mathbf{x}_C(s,t) + \mathbf{R}(s,t)\left(\bar{\mathbf{r}}(c) - \bar{\mathbf{r}}_C\right) + \mathbf{R}(s,t)\,\mathbf{w}(s,c,t) \qquad [7.33]$$

where $\mathbf{w}(s,c,t)$ is a "distortional" vector field. As a result, the displacement $\mathbf{u} := \mathbf{x}(s,\bar{\mathbf{r}}(c),t) - \bar{\mathbf{x}}(s,\bar{\mathbf{r}}(c))$ of the point P reads[12]:

$$\mathbf{u} = \mathbf{u}_C(s,t) + \left(\mathbf{R}(s,t) - \mathbf{I}\right)\left(\bar{\mathbf{r}}(c) - \bar{\mathbf{r}}_C\right) + \mathbf{R}(s,t)\,\mathbf{w}(s,c,t) \qquad [7.34]$$

Such a representation, however, would lead to a 2D-model, due to the dependence of $\mathbf{w}$ on two spatial coordinates. Therefore, in order to reduce them to one, and exploiting the basic idea of GBT, we express the distortion $\mathbf{w}$ as a linear combination of *known functions* ψ's of the directrix-abscissa, and *unknown functions* a's of the axis-abscissa (in addition to time), such that:

$$\mathbf{w} = \sum_{j=1}^{n} a_j(s,t)\,\bar{\boldsymbol{\psi}}_j(c) \qquad [7.35]$$

where a_j are *distortional amplitudes* and:

$$\bar{\boldsymbol{\psi}}_j := \psi_{tj}(c)\,\bar{\mathbf{a}}_t(c) + \psi_{nj}(c)\,\bar{\mathbf{a}}_n(c) + \psi_{wj}(c)\,\bar{\mathbf{a}}_1 \qquad [7.36]$$

are *distortional modes*, whose components onto the intrinsic basis are ψ_{tj}, ψ_{nj} (describing the in-plane change of shape of the cross-section) and ψ_{wj} (describing the out-of-plane displacement, called *warping*).

The distortional modes, the translation and the rotation must be a set of *linearly independent* functions, in the sense that none of them must be a linear combination of the remaining ones. This property assures that distortions are *not rigid* motions, and that any of them describes essentially different ways in which the cross-section loses its initial shape.

11. Note the slight change of notations with respect to the 1D-model. Now, the old $\mathbf{x}$ and $\mathbf{u}$ are changed into $\mathbf{x}_C$ and $\mathbf{u}_C$, since the new $\mathbf{x}$ and $\mathbf{u}$ are referred to the generic point $P \neq G$.

12. Indeed, $\bar{\mathbf{x}} = \bar{\mathbf{x}}_C + (\bar{\mathbf{r}} - \bar{\mathbf{r}}_C)$ (see Figure 7.1).

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Fiber-models and local strains

With equations [7.35] and [7.36], the equation [7.34] constitutes a *displacement-map*, of the form [7.32a]. To build up the strain-map, we need to introduce local measures for the strains. The simplest way to proceed would be using a 3D strain tensor for Cauchy continuum, for example the Green–Lagrange tensor, as done in [DI 03a], and several other works. This approach, however, has the drawback of leading to expressions for local strains which do not *exclusively* involve generalized strains for the 1D-model, i.e. are *not* of the form [7.32b], and therefore are *not* suited to identify the 1D-model. Therefore, a suitable 3D-model has to be introduced, along the lines we followed in Chapter 2 by dealing with compact beams, but accounting, at the same time, for the loss of shape of the cross-section.

To this purpose, ideally we can buildup a physical model of TWB, as made of a *bundle of thin rods* (or fibers), transversely connected by an infinite number of *ribs*, where each element of the bundle complies with the kinematic laws that we established for the beam as a whole. We consider three different cases.

1) *If the ribs are infinitely rigid*, both in their plane and out-of-plane, then the TWB is also locally rigid, since the cross-section is constrained to remain planar. In this case, the unique strains allowed are those of the rods, i.e. local extension, shear-strains, flexural and torsional curvatures. However, if we admit that the distortion is weakly variable along the beam axis (i.e. if we exclude deformation patterns like those occurring in local buckling), then the local flexure of the rods is expected to be negligible in comparison with the extension of the same fibers. Local torsion, instead, is of some interest, since it allows us to capture the de Saint-Venant torsion mechanism, that otherwise would be lost, after the cross-section has been flattened on its middle-line.

2) *If the ribs are infinitely rigid in their plane, but infinitely flexible out-of-plane*, then local deformability only concerns warping (according to the fundamental Vlasov hypothesis on which the classical TWB theory is grounded, [VLA 61]). In this case, strains are as before, but, due to the loss of planarity of the cross-section, the local normal $\mathbf{a}_w(c)$ at the deformed cross-section at the abscissa c now depends on c, and it is no more parallel to the binormal $\mathbf{a}_b$ to the unwarped section.

3) Finally, *if the ribs have finite in-plane stiffness* and zero out-of-plane stiffness, then local deformability also concerns the loss of shape of the cross-section. Strains of the fibers are no more sufficient to describe the state of strain of the TWB, but extension and flexure of the ribs must be accounted for, as planar curved beam.

Case (1) has been addressed in Chapter 2; cases (2) and (3) will be considered further on.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Working plan

A general treatment of a fully deformable TWB is, unfortunately, far beyond the scope of this book, since the complex topic would require a dedicated volume. Therefore, in the next few sections, we will confine ourselves to two comparatively simple models:

1) warpable TWB model, with cross-sections undeformable in their own plane;

2) planar, unwarpable, TWB model, with cross-sections deformable in their own plane.

To keep the algebra as simple as possible, we will account for distortion by means of a *unique mode*, of amplitude a (i.e. we take $n = 1$ in equation [7.35]) by leaving the quite obvious extension to several parameters to the reader. Accordingly, we have (a) $\bar{\boldsymbol{\psi}} = \psi_w(c)\,\bar{\mathbf{a}}_1$ or (b), $\bar{\boldsymbol{\psi}} = \psi_t(c)\,\bar{\mathbf{a}}_t(c) + \psi_n(c)\,\bar{\mathbf{a}}_n(c)$. Since warping is mainly due to torsion[13], we will refer case (b) to planar beams.

7.6 Warpable, cross-undeformable TWB

We consider a fiber-model of TWB, whose cross-section is rigid in its own plane, but it is free to warp out-of-plane. The cross-section can indifferently be open or closed. Moreover, we assume that the beam is unshearable.

7.6.1 Kinematics

Displacement field

From equations [7.33], [7.35] and [7.36], it follows that the position field of a TWB only undergoing a warping distortion is:

$$\mathbf{x} = \mathbf{x}_C(s,t) + \mathbf{R}(s,t)\left(\bar{\mathbf{r}}(c) - \bar{\mathbf{r}}_C\right) + a_w(s,t)\,\psi_w(c)\,\mathbf{R}(s,t)\,\bar{\mathbf{a}}_1 \qquad [7.37]$$

where $\psi_w(c)$ is the warping function. As a result:

$$\mathbf{u} = \mathbf{u}_C(s,t) + \left(\mathbf{R}(s,t) - \mathbf{I}\right)\left(\bar{\mathbf{r}}(c) - \bar{\mathbf{r}}_C\right) + a_w(s,t)\,\psi_w(c)\,\mathbf{R}(s,t)\,\bar{\mathbf{a}}_1 \qquad [7.38]$$

is a *displacement-map* of the form [7.32a].

In order to assure that $\psi_w(c)$ is a pure distortion, it has to be neither a translation nor a rotation. To formalize this property, we will require that the distortion mode is

13. Warping due to shear strains is usually approximately accounted for via shear-factors.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

"orthogonal" to *any* translation along $\bar{\mathbf{a}}_1$ and *any* rotation around an axis contained in the plane of the cross-section. Since the latter are linear in r_2, r_3, we have:

$$\int_{\mathcal{C}} \psi_w(c)\, b dc = 0, \qquad \int_{\mathcal{C}} r_2(c)\, \psi_w(c)\, b dc = 0$$
$$\int_{\mathcal{C}} r_3(c)\, \psi_w(c)\, b dc = 0 \qquad [7.39]$$

These conditions will be referred to as *orthogonality conditions*[14]. The warping function can, in principle, be chosen arbitrarily, provided it is sufficiently smooth.

Longitudinal and transverse local strains

Because of warping, the cross-section is no longer planar in the actual configuration, but it is an (assumed) smooth surface. It is useful to evaluate the *local normal* $\bar{\mathbf{a}}_w = \bar{\mathbf{a}}_w(s, c, t)$ to this surface, at the generic point P on $\mathcal{C}$, when the surface is pulled back to the reference configuration. It turns out that[15]:

$$\bar{\mathbf{a}}_w = \bar{\mathbf{a}}_1 - a_w(s, t)\, \frac{d\psi_w(c)}{dc} \bar{\mathbf{a}}_t \qquad [7.40]$$

We define the local (right) strain vector $\mathbf{e}(s, c, t)$ as *the difference between the tangent to the longitudinal fiber passing for P, and the local normal to the warped cross-section*, both *pulled back* to the reference configuration, i.e.:

$$\mathbf{e} = \mathbf{R}^T(s, t)\, \mathbf{x}'(s, t) - \bar{\mathbf{a}}_w(s, c, t) \qquad [7.41]$$

It should be noted that this expression correctly reduces to zero, if the transformation is rigid (i.e. if $\mathbf{R}^T\mathbf{x}' = \bar{\mathbf{a}}_1$ and $\psi_w \equiv 0$). If we make use of equations [7.37] and

14. Orthogonality is a stronger condition than linear independence. It is not strictly necessary, but sometimes simplifies the resulting expression. In some other cases, in contrast, it can be more conveniently relaxed in a "not-parallelism condition". To better understand the concept, let us consider a discrete system, made of two masses linked by an elastic spring and free to move along a straight line. The matrix column $(1, 1)$ represents a rigid motion, while $(-1, 1)$ is a purely distortional mode, satisfying orthogonality. However, $(1, 0)$ and $(0, 1)$ are also possible distortional modes, although they are non-orthogonal to the rigid motion.

15. Note that $\|\bar{\mathbf{a}}_w\| = 1 + \mathrm{O}\left(a_w^2\right)$, so that $\bar{\mathbf{a}}_w$ should be normalized to 1. However, we will avoid doing so, since, however, it leads to an *exact measure of strain* (the next equation [7.41]). Therefore, normalization would only modify the physical meaning of the strain, to within the usual error we make confusing longitudinal and transverse strains with unit extension and shear-strains.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[7.40], the strain assumes the form:

$$\begin{aligned}\mathbf{e} &= \mathbf{R}^T\left[\mathbf{x}'_C + \mathbf{R}'\left(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C\right) + \psi_w\left(\mathbf{R}' a_w + \mathbf{R} a'_w\right)\bar{\mathbf{a}}_1\right] \\ &\quad - \bar{\mathbf{a}}_1 + a_w \frac{d\psi_w}{dc}\bar{\mathbf{a}}_t \\ &= \left(\mathbf{R}^T\mathbf{x}'_C - \bar{\mathbf{a}}_1\right) + \mathbf{R}^T\mathbf{R}'\left(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C\right) \\ &\quad + \psi_w\left(\mathbf{R}^T\mathbf{R}' a_w + a'_w\right)\bar{\mathbf{a}}_1 + a_w\frac{d\psi_w}{dc}\bar{\mathbf{a}}_t\end{aligned} \qquad [7.42]$$

which can be *expressed in terms of the sole generalized strains of the 1D-model*, such that:

$$\mathbf{e} = \mathbf{e}_C + \mathbf{k} \times \left(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C + \alpha_w\psi_w\bar{\mathbf{a}}_1\right) + \alpha_w\frac{d\psi_w}{dc}\bar{\mathbf{a}}_t + \beta_w\psi_w\bar{\mathbf{a}}_1 \qquad [7.43]$$

where:

$$\alpha_w := a_w, \qquad \beta_w := a'_w \qquad [7.44]$$

are the *warping strain* and the *warping strain gradient*, respectively.

Equation [7.43] defines the local strains as the sum of four contributions: (a) the strain at the flexural-axis; (b) the strain due to the beam curvature; this is similar to that which we found for unwarpable beams (equations [2.159] and [2.174]), but it accounts now for the fact that the planar "arm" $\bar{\mathbf{r}} - \bar{\mathbf{r}}_C$ is increased by an out-of-plane component, caused by warping, which brings a second-order contribution to the strain; (c) a transverse-strain, induced by the variation of warping along the middle-line $\mathcal{C}$; and (d) a longitudinal strain, induced by the variation of warping along the beam axis.

By decomposing all the vectors along the plane $\pi := \mathrm{span}\left(\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3\right)$ and its normal $\bar{\mathbf{a}}_1$, i.e. by using :

$$\mathbf{e} = \varepsilon\bar{\mathbf{a}}_1 + \boldsymbol{\gamma}, \qquad \mathbf{e}_C = \varepsilon_C\bar{\mathbf{a}}_1 + \boldsymbol{\gamma}_C, \qquad \mathbf{k} = \kappa_1\bar{\mathbf{a}}_1 + \mathbf{k}_\pi \qquad [7.45]$$

we have:

$$\begin{aligned}\varepsilon &= \varepsilon_C + \mathbf{k}_\pi \times \left(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C\right)\cdot\bar{\mathbf{a}}_1 + \beta_w\psi_w \\ \boldsymbol{\gamma} &= \boldsymbol{\gamma}_C + \kappa_1\bar{\mathbf{a}}_1 \times \left(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C\right) + \alpha_w\left(\mathbf{k}_\pi \times \bar{\mathbf{a}}_1\psi_w + \frac{d\psi_w}{dc}\bar{\mathbf{a}}_t\right)\end{aligned} \qquad [7.46]$$

We now introduce the hypothesis that the *beam is unshearable*. Since the cross-section loses its planarity, we have to specify *which fiber* of the bundle remains collinear to the local normal to the warped cross-section. Being the shear referred to the flexural center, it appears natural to select just this axis, by requiring that $\boldsymbol{\gamma}_C = \mathbf{0}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The transverse strain can be projected further onto the tangent and onto the normal to the middle-line; by letting $\gamma_t := \boldsymbol{\gamma} \cdot \bar{\mathbf{a}}_t$, $\gamma_n := \boldsymbol{\gamma} \cdot \bar{\mathbf{a}}_n$, we have:

$$\begin{aligned} \gamma_t &= \kappa_1 \left(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C\right) \times \bar{\mathbf{a}}_t \cdot \bar{\mathbf{a}}_1 + \alpha_w \left(\mathbf{k}_\pi \times \bar{\mathbf{a}}_1 \cdot \bar{\mathbf{a}}_t \psi_w + \frac{d\psi_w}{dc}\right) \\ \gamma_n &= \kappa_1 \left(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C\right) \times \bar{\mathbf{a}}_n \cdot \bar{\mathbf{a}}_1 + \alpha_w \mathbf{k}_\pi \times \bar{\mathbf{a}}_1 \cdot \bar{\mathbf{a}}_n \psi_w \end{aligned} \quad [7.47]$$

According to a commonly accepted hypothesis, we will neglect the shear-strains γ_n normal to the middle-line $\mathcal{C}$. Indeed, a full 3D analysis would require them to vanish at the external and internal boundaries (i.e. at the ends of the chord Ξ), and, since the thickness is small, it is reasonable to assume they vanish everywhere[16].

Local curvatures

The local curvatures of the fibers do *not* coincide with that of the beam as a whole, since warping modifies the local triad intrinsic to the warped cross-section. However, warping causes rotation of the local normal around $\mathbf{a}_n$, so that it only affects the flexural curvatures. The latter, as we observed, are negligible, since they bring a small contribution to the elastic energy, and therefore we will only account for the local twist, κ_t, equal to the global twist, namely:

$$\kappa_t = \kappa_1 \quad [7.48]$$

Summary of local strains

By using equation [7.21a] in equation [7.46a], and appending equations [7.46b] and [7.48], we finally get:

$$\begin{aligned} \varepsilon &= \varepsilon_G + \mathbf{k}_\pi \times \bar{\mathbf{r}} \cdot \bar{\mathbf{a}}_1 + \beta_w \psi_w \\ \gamma_t &= \kappa_1 \left(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C\right) \times \bar{\mathbf{a}}_t \cdot \bar{\mathbf{a}}_1 + \alpha_w \left(\mathbf{k}_\pi \times \bar{\mathbf{a}}_1 \cdot \bar{\mathbf{a}}_t \psi_w + \frac{d\psi_w}{dc}\right) \\ \kappa_t &= \kappa_1 \end{aligned} \quad [7.49]$$

This is a *strain-map* of the form [7.32b].

16. A more refined analysis, in which the variation of the warping along the chord is accounted for, would make it possible to remove the linear part of γ_n, in the same way it occurs in the de Saint-Venant problem of uniform torsion.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Velocity and strain-rate field

To evaluate the velocity field, we time-differentiate equations [7.37]; since:

$$\left(a_w\left(s,t\right)\mathbf{R}\left(s,t\right)\bar{\mathbf{a}}_1\right)^{\cdot}=\left(\dot{a}_w\mathbf{R}+a_w\dot{\mathbf{R}}\right)\bar{\mathbf{a}}_1=\left(\dot{a}_w+a_w\dot{\mathbf{R}}\mathbf{R}^T\right)\mathbf{a}_1 \qquad [7.50]$$

we have:

$$\begin{aligned}\mathbf{v}&=\mathbf{v}_C\left(s,t\right)+\boldsymbol{\omega}\left(s,t\right)\times\left(\mathbf{r}-\mathbf{r}_C\right)\\&\quad+\left(\dot{a}_w\left(s,t\right)\mathbf{a}_1+a_w\left(s,t\right)\boldsymbol{\omega}\left(s,t\right)\times\mathbf{a}_1\right)\psi_w\left(c\right)\end{aligned} \qquad [7.51]$$

in which the *warping velocity* $\dot{a}_w$ appears. This is a *velocity-map* of the form [7.32c]. Note that the velocity field is referred to the flexural center, consistently with displacements.

The strain-rate velocity field follows from time-differentiation of equations [7.49]:

$$\begin{aligned}\dot{\varepsilon}&=\dot{\varepsilon}_G+\dot{\mathbf{k}}_\pi\times\bar{\mathbf{r}}\cdot\bar{\mathbf{a}}_1+\dot{\beta}_w\psi_w\\\dot{\gamma}_t&=\dot{\kappa}_1\left(\bar{\mathbf{r}}-\bar{\mathbf{r}}_C\right)\times\bar{\mathbf{a}}_t\cdot\bar{\mathbf{a}}_1\\&\quad+\alpha_w\dot{\mathbf{k}}_\pi\times\bar{\mathbf{a}}_1\cdot\bar{\mathbf{a}}_t\psi_w+\dot{\alpha}_w\left(\mathbf{k}_\pi\times\bar{\mathbf{a}}_1\cdot\bar{\mathbf{a}}_t\psi_w+\frac{d\psi_w}{dc}\right)\\\dot{\kappa}_t&=\dot{\kappa}_1\end{aligned} \qquad [7.52]$$

or:

$$\begin{aligned}\dot{\varepsilon}&=\dot{\varepsilon}_G+\mathbf{R}\dot{\mathbf{k}}_\pi\times\mathbf{r}\cdot\mathbf{a}_1+\dot{\beta}_w\psi_w\\\dot{\gamma}_t&=\dot{\kappa}_1\left(\mathbf{r}-\mathbf{r}_C\right)\times\mathbf{a}_t\cdot\mathbf{a}_1\\&\quad+\alpha_w\mathbf{R}\dot{\mathbf{k}}_\pi\times\mathbf{a}_1\cdot\mathbf{a}_t\psi_w+\dot{\alpha}_w\left(\mathbf{R}\mathbf{k}_\pi\times\mathbf{a}_1\cdot\mathbf{a}_t\psi_w+\frac{d\psi_w}{dc}\right)\\\dot{\kappa}_t&=\dot{\kappa}_1\end{aligned} \qquad [7.53]$$

where all the vector factors of the mixed products have been rotated. A *strain-rate-map* of the form [7.32d] is thus obtained.

7.6.2 Identification procedure

Generalized external force identification

To identify external forces, the external virtual powers of the two models are equated. To this end, we consider body forces $\mathbf{b}\left(s,\mathbf{r},t\right)$ (having the dimension of forces per unit of volume $\left[\mathrm{ML}^{-2}\mathrm{T}^{-2}\right]$) applied to the volume $\mathcal{V}$ of the cylinder, and surface forces $\mathbf{f}_H\left(\mathbf{r},t\right)$ (having the dimension of forces per unit of

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

surface $\left[\mathrm{ML}^{-1}\mathrm{T}^{-2}\right]$) applied to the bases $\mathcal{A}_H$, with $H = A, B$. Then, we superimpose an admissible virtual motion $\mathbf{v}(s, \mathbf{r}, t)$ to the current configuration of the beam, and evaluate the external virtual power spent by the external forces, i.e.:

$$\begin{aligned}\mathcal{P}_{ext} &= \int_{\mathcal{V}} \mathbf{b} \cdot \mathbf{v} dV + \sum_{H=A}^{B} \int_{\mathcal{A}_H} \mathbf{f}_H \cdot \mathbf{v}_H dA \\ &= \int_{\mathcal{S}} ds \int_{\mathcal{C}} \mathbf{b} \cdot [\mathbf{v}_C + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_C) \\ &+ (\dot{a}_w \mathbf{a}_1 + a_w \boldsymbol{\omega} \times \mathbf{a}_1)\, \psi_w] bdc \\ &+ \sum_{H=A}^{B} \int_{\mathcal{A}_H} \mathbf{f}_H \cdot [\mathbf{v}_{CH} + \boldsymbol{\omega}_H \times (\mathbf{r} - \mathbf{r}_C) \\ &+ (\dot{a}_{wH} \mathbf{a}_1 + a_{wH} \boldsymbol{\omega}_H \times \mathbf{a}_1)\, \psi_w] bdc\end{aligned} \qquad [7.54]$$

in which $dA = bdc$ is the area element, $dV = dAds$ is the volume element and the velocity-map [7.51] has been used. Since the external virtual power [7.22] for the 1D-model reads:

$$\begin{aligned}\mathcal{P}_{ext} &:= \int_{\mathcal{S}} (\mathbf{p} \cdot \mathbf{v}_C + \mathbf{c} \cdot \boldsymbol{\omega} + q_w \dot{a}_w)\, ds \\ &+ \sum_{H=A}^{B} (\mathbf{P}_H \cdot \mathbf{v}_{CH} + \mathbf{C}_H \cdot \boldsymbol{\omega}_H + Q_{wH} \dot{a}_{wH})\end{aligned} \qquad [7.55]$$

the following generalized forces are identified:

$$\begin{aligned}\mathbf{p} &:= \int_{\mathcal{C}} \mathbf{b} bdc, & \mathbf{c} &:= \int_{\mathcal{C}} (\mathbf{r} - \mathbf{r}_C + a_w \psi_w \mathbf{a}_1) \times \mathbf{b} bdc \\ \mathbf{P}_H &:= \int_{\mathcal{C}_H} \mathbf{f}_H bdc, & \mathbf{C}_H &:= \int_{\mathcal{C}_H} (\mathbf{r} - \mathbf{r}_C + a_{wH} \psi_w \mathbf{a}_1) \times \mathbf{f}_H bdc \\ q_w &:= \int_{\mathcal{C}} \psi_w \mathbf{a}_1 \cdot \mathbf{b} bdc, & Q_{wH} &:= \int_{\mathcal{C}_H} \psi_w \mathbf{a}_1 \cdot \mathbf{f}_H bdc\end{aligned} \qquad [7.56]$$

The forces $\mathbf{p}, \mathbf{P}_H$ are the resultants of the body- and surface-forces. Similarly, the couples $\mathbf{c}, \mathbf{C}_H$ are the resultant moments, which, however, account for the change of geometry induced by the warping. The forces q_w and Q_{wH} are the scalar products of the body- and surface-forces by the warping function. It appears that they are of the same type of the other forces, if one replaces the "rigid" weight functions $1, r_2(c), r_3(c)$ by the "flexible" weight function $\psi_w(c)$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Generalized stress identification

To identify generalized stresses, the internal virtual power of the two models is equated. By denoting by $\boldsymbol{\sigma} = \sigma \mathbf{a}_1 + \tau_t \mathbf{a}_t$ the stresses, and by m_t the local torsional moment (having the dimension of a couple for surface unit, $[\mathrm{MT}^{-2}]$), all acting in the current configuration, the internal virtual power of the 3D-model reads:

$$
\begin{aligned}
\mathcal{P}_{int} &= \int_{\mathcal{V}} (\sigma \dot{\varepsilon} + \tau_t \dot{\gamma}_t + m_t \dot{\kappa}_t)\, dV \\
&= \int_{\mathcal{S}} ds \int_{\mathcal{C}} \Big[\sigma \left(\dot{\varepsilon}_G + \mathbf{r} \times \mathbf{a}_1 \cdot \mathbf{R}\dot{\mathbf{k}}_\pi \right) \\
&+ \tau_t \left(\dot{\kappa}_1 (\mathbf{r} - \mathbf{r}_C) \times \mathbf{a}_t \cdot \mathbf{a}_1 \right) + m_t \dot{\kappa}_1 \Big] bdc \\
&+ \int_{\mathcal{S}} ds \int_{\mathcal{C}} \Big[\sigma \dot{\beta}_w \psi_w + \tau_t \Big(\dot{\alpha}_w \left(\mathbf{R}\mathbf{k}_\pi \times \mathbf{a}_1 \cdot \mathbf{a}_t \psi_w + \frac{d\psi_w}{dc} \right) \\
&+ \alpha_w \mathbf{a}_1 \times \mathbf{a}_t \cdot \mathbf{R}\dot{\mathbf{k}}_\pi \psi_w \Big) \Big] bdc
\end{aligned}
\qquad [7.57]
$$

where the strain-rate-map [7.53] has been used. Since the internal virtual power [7.23], for the 1D-model, after introducing shear-undeformability, reads:

$$
\mathcal{P}_{int} := \int_{\mathcal{S}} \left(N \dot{\varepsilon}_G + M_1 \dot{\kappa}_1 + \mathbf{m}_\pi \cdot \mathbf{R}\dot{\mathbf{k}}_\pi + D_w \dot{\alpha}_w + B_w \dot{\beta}_w \right) ds \qquad [7.58]
$$

the generalized active stresses are identified as follows:

$$
\begin{aligned}
N &= \int_{\mathcal{C}} \sigma bdc, \qquad M_1 = \int_{\mathcal{C}} \left[(\mathbf{r} - \mathbf{r}_C) \times \tau_t \mathbf{a}_t \cdot \mathbf{a}_1 + m_t \right] bdc \\
\mathbf{m}_\pi &:= \int_{\mathcal{C}} (\mathbf{r} \times \sigma \mathbf{a}_1 + \alpha_w \psi_w \mathbf{a}_1 \times \tau_t \mathbf{a}_t)\, bdc \\
D_w &= \int_{\mathcal{C}} \tau_t \left(\mathbf{R}\mathbf{k}_\pi \cdot \mathbf{a}_n \psi_w + \frac{d\psi_w}{dc} \right) bdc, \qquad B_w := \int_{\mathcal{C}} \sigma \psi_w bdc
\end{aligned}
\qquad [7.59]
$$

The planar component $\mathbf{t}_\pi$ of the internal force (shear-force) is instead of a reactive type.

The first three of the active stresses can be interpreted as the resultant of the local stresses, as for the locally rigid beam (with the flexural moment evaluated with respect to the centroid, and the torsional moment with respect to the flexural

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

center[17]). However, since the cross-section is warped in the current configuration, even the tangential stress contributes to the flexural moment. Moreover, the distortional stresses represent the virtual power of the local stresses in the local strain-rates produced by warping.

Elastic constant identification

To identify the constitutive law for hyperelastic material, the elastic potential of the two models must be equated. By assuming that the material is linearly elastic, the potential for the 3D-model, relevant to the whole length of the beam, reads:

$$\begin{aligned} U &:= \int_{\mathcal{S}} ds \int_{\mathcal{C}} \frac{1}{2}\left(E\varepsilon^2 + G\gamma_t^2 + \frac{1}{3}Gb^2\kappa_t^2\right) bdc \\ &=: \int_{\mathcal{S}} \left(\phi_\varepsilon + \phi_\gamma + \phi_{\kappa_t}\right) ds \end{aligned} \qquad [7.60]$$

where E is the Young modulus, G is the tangential modulus and $Gb^2/3$ is the de Saint-Venant torsional stiffness per unit of area[18]. By using the strain-map [7.49], we observe that, due to the geometrical properties of the principal centroid-axis and the orthogonality conditions [7.39]:

$$\begin{aligned} \phi_\varepsilon &= \frac{1}{2}E\int_{\mathcal{C}} \left(\varepsilon_G + \kappa_2 r_3 - \kappa_3 r_2 + \beta_w\psi_w\right)^2 bdc \\ &= \frac{1}{2}\left(EA\varepsilon_G^2 + EJ_2\kappa_2^2 + EJ_3\kappa_3^2 + E\Gamma_w\beta_w^2\right) \\ \phi_{\kappa_t} &= \frac{1}{2}GJ_{SV}\kappa_1^2 \\ \phi_\gamma &= \frac{1}{2}G\int_{\mathcal{C}} \left(\kappa_1 r_{nC} + \alpha_w\frac{d\psi_w}{dc} + \alpha_w\mathbf{k}_\pi\cdot\bar{\mathbf{a}}_n\psi_w\right)^2 bdc \\ &= \frac{1}{2}G\left(J_n\kappa_1^2 + J_w\alpha_w^2 + \alpha_w^2\mathbf{k}_\pi\cdot\boldsymbol{\Gamma}_w\mathbf{k}_\pi\right) \\ &\quad + G\left(J_{nw}\kappa_1\alpha_w + \kappa_1\alpha_w\mathbf{k}_\pi\cdot\mathbf{y}_n + \alpha_w^2\mathbf{k}_\pi\cdot\mathbf{y}_w\right) \end{aligned} \qquad [7.61]$$

17. In the torsional moment, the local contribution m_t accounts for the variation of the tangential stresses along the chord.

18. Note that this stiffness is relevant to an *open* flow, as for a thin rectangular section.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where the following geometric characteristics have been introduced[19]:

$$
\begin{aligned}
&A := \int_{\mathcal{C}} b dc, && J_2 := \int_{\mathcal{C}} r_3^2 b dc, && J_3 := \int_{\mathcal{C}} r_2^2 b dc \\
&J_n := \int_{\mathcal{C}} r_{nC}^2 b dc, && J_{nw} := \int_{\mathcal{C}} r_{nC} \frac{d\psi_w}{dc} b dc, && J_w := \int_{\mathcal{C}} \left(\frac{d\psi_w}{dc}\right)^2 b dc \\
&J_{SV} := \frac{1}{3}\int_{\mathcal{C}} b^3 dc, && \mathit{\Gamma}_w := \int_{\mathcal{C}} \psi_w^2 b dc, && \mathbf{\Gamma}_w := \int_{\mathcal{C}} \bar{\mathbf{a}}_n \otimes \bar{\mathbf{a}}_n \psi_w^2 b dc \\
&\mathbf{y}_n := \int_{\mathcal{C}} r_{nc}\psi_w \bar{\mathbf{a}}_n b dc, && \mathbf{y}_w := \int_{\mathcal{C}} \psi_w \frac{d\psi_w}{dc} \bar{\mathbf{a}}_n b dc
\end{aligned}
\tag{7.62}
$$

in which $\mathit{\Gamma}_w$ is the *warping torsional stiffness* and J_{SV} is the *de Saint-Venant torsional stiffness* of open TWB; moreover $r_{nC} := (\bar{\mathbf{r}}(c) - \bar{\mathbf{r}}_C) \times \bar{\mathbf{a}}_t \cdot \bar{\mathbf{a}}_1 = -(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C) \cdot \bar{\mathbf{a}}_n$ is the normal distance from C of a point P on $\mathcal{C}$. Furthermore, $\mathbf{y}_n, \mathbf{y}_w$ are constant vectors, having components Y_{ni}, Y_{wi}, and $\mathbf{\Gamma}_w$ is a constant symmetric tensor, of components $\mathit{\Gamma}_{wij}$, with $i, j = 2, 3$[20].

By requiring that $U = \frac{1}{2}\int_{\mathcal{C}} \phi ds$, we get the elastic potential for the 1D-model:

$$
\begin{aligned}
\phi = \frac{1}{2}[&EA\varepsilon_G^2 + (GJ_{SV} + GJ_n)\kappa_1^2 + EJ_2\kappa_2^2 + EJ_3\kappa_3^2 + GJ_w\alpha_w^2 \\
&+ E\mathit{\Gamma}_w\beta_w^2 + 2GJ_{nw}\kappa_1\alpha_w] \\
&+ G\kappa_1\alpha_w\mathbf{k}_\pi \cdot \mathbf{y}_n + G\alpha_w^2\mathbf{k}_\pi \cdot \mathbf{y}_w + \frac{1}{2}G\alpha_w^2\mathbf{k}_\pi \cdot \mathbf{\Gamma}_w\mathbf{k}_\pi
\end{aligned}
\tag{7.63}
$$

This expression, however, suffers the drawbacks we already discussed about the locally rigid beam (section 2.3.2), i.e. it is unable to capture the "shortening effect" due to the torsion–extension and torsion–flexure couplings. By proceeding as in that

19. In attributing a name to each quantity, we thought ψ_w to be an area $[\mathrm{L}^2]$, according to the meaning it takes in the Vlasov theory (see section 8.2.1 ahead). As a result, $[J_\alpha] = [\mathrm{L}^4]$, $[Y_\alpha] = [\mathrm{L}^5]$ and $[\mathit{\Gamma}_\alpha] = [\mathrm{L}^6]$.

20. By letting $\bar{\mathbf{a}}_n = -\sin\varphi\bar{\mathbf{a}}_2 + \cos\varphi\bar{\mathbf{a}}_3$ (in which $\varphi(c)$ is the angle in figure 7.1(b)), the scalar representation of the tensor product $\bar{\mathbf{a}}_n \otimes \bar{\mathbf{a}}_n$ is:

$$
\begin{bmatrix} \sin^2\varphi & -\sin\varphi\cos\varphi \\ -\sin\varphi\cos\varphi & \cos^2\varphi \end{bmatrix}
$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

case, we correct it, by adding cubic terms (although inconsistently) deriving from the sum of $\frac{1}{2}\kappa_1^2 (r - r_C)^2$ to the longitudinal strain [7.49a], thus obtaining:

$$\begin{aligned}\phi = \frac{1}{2}&[EA\varepsilon_G^2 + (GJ_{SV} + GJ_n)\,\kappa_1^2 + EJ_2\kappa_2^2 + EJ_3\kappa_3^2 \\ &+ GJ_w\alpha_w^2 + E\Gamma_w\beta_w^2 + 2GJ_{nw}\kappa_1\alpha_w] \\ &+ G\kappa_1\alpha_w\mathbf{k}_\pi\cdot\mathbf{y}_n + G\alpha_w^2\mathbf{k}_\pi\cdot\mathbf{y}_w + \frac{1}{2}G\alpha_w^2\mathbf{k}_\pi\cdot\mathbf{\Gamma}_w\mathbf{k}_\pi \\ &+ \frac{1}{2}\left(EJ_C\varepsilon_G\kappa_1^2 - EY_{3C}\kappa_3\kappa_1^2 + EY_{2C}\kappa_2\kappa_1^2 + E\Gamma_{wC}\beta_w\kappa_1^2\right)\end{aligned} \qquad [7.64]$$

where (remember equation [2.176]):

$$\begin{aligned}J_C &:= \int_{\mathcal{C}} (r - r_C)^2\, bdc, \qquad &\Gamma_{wC} &:= \int_{\mathcal{C}} \psi_w\,(r - r_C)^2\, bdc \\ Y_{2C} &:= \int_{\mathcal{C}} r_3\,(r - r_C)^2\, bdc, \qquad &Y_{3C} &:= \int_{\mathcal{C}} r_2\,(r - r_C)^2\, bdc\end{aligned} \qquad [7.65]$$

From the potential [7.64] the following constitutive law is drawn:

$$\begin{pmatrix} N \\ M_1 \\ M_2 \\ M_3 \\ D_w \\ B_w \end{pmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & GJ_{SV} + GJ_n & 0 & 0 & GJ_{nw} & 0 \\ 0 & 0 & EJ_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & EJ_3 & 0 & 0 \\ 0 & GJ_{nw} & 0 & 0 & GJ_w & 0 \\ 0 & 0 & 0 & 0 & 0 & E\Gamma_w \end{bmatrix} \begin{pmatrix} \varepsilon_G \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \alpha_w \\ \beta_w \end{pmatrix} + \boldsymbol{f}(\boldsymbol{\varepsilon}) \qquad [7.66]$$

where:

$$\boldsymbol{f}(\boldsymbol{\varepsilon}) := \begin{pmatrix} \frac{1}{2}EJ_C\kappa_1^2 \\ \kappa_1\left(EJ_C\varepsilon_G - EY_{3C}\kappa_3 + EY_{2C}\kappa_2\right) \\ \frac{1}{2}EY_{2C}\kappa_1^2 \\ -\frac{1}{2}EY_{3C}\kappa_1^2 \\ G\kappa_1\mathbf{k}_\pi\cdot\mathbf{y}_n \\ \frac{1}{2}E\Gamma_{wC}\kappa_1^2 \end{pmatrix} + \begin{pmatrix} 0 \\ E\Gamma_{wC}\beta_w\kappa_1 \\ GY_{n2}\alpha_w\kappa_1 + \alpha_w^2\left(GY_{w2} + G\Gamma_{w22}\kappa_2 + G\Gamma_{w23}\kappa_3\right) \\ GY_{n3}\alpha_w\kappa_1 + \alpha_w^2\left(GY_{w3} + G\Gamma_{w32}\kappa_2 + G\Gamma_{w33}\kappa_3\right) \\ \alpha_w\left(G\mathbf{k}_\pi\cdot\mathbf{y}_w + G\mathbf{k}_\pi\cdot\mathbf{\Gamma}_w\mathbf{k}_\pi\right) \\ 0 \end{pmatrix} \qquad [7.67]$$

is the nonlinear part.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

REMARK 7.3. The linear part of the elastic law [7.66d] states that the torsional moment depends not only on the torsional curvature κ_1, but also on the warping α_w. Torsion κ_1 brings two contributions: (a) one is due to the local twist of the fibers, leading to the de Saint-Venant stiffness GJ_{SV} for TWB; (b) the other is due to the shear-strain induced by a twist of the section around the flexural center (remember equation [7.46b]), expressed by the inertia stiffness GJ_n. Warping α_w also brings a further contribution, via the stiffness GJ_{nw}. Since the linear torsional stiffness of a TWB is just GJ_{SV}, it is expected that the two additional terms cancel each other out, when a suitable warping function is chosen. This is indeed the case, as we will show later, in section 8.3.

7.6.3 The Fundamental Problem

The Fundamental Problem for the 1D, unshearable, warpable and cross-undeformable TWB is governed by the following equations, when warping is described by just one distortion variable a_w:

1) six strain–displacement relationships [7.16a,d,e,f] and [7.44];

2) two unshearability conditions, derived from equations [7.16b,c] by zeroing γ_{2C}, γ_{3C};

3) seven balance equations [7.26] (with index j replaced by w);

4) six elastic law [7.66], linking the active stresses to the admissible strains.

The mechanical boundary conditions are stated by equations [7.27] (with index j replaced by w).

The previous equations express the mixed formulation for the internally constrained TWB. They are a system of 21 equations in the following unknowns: (a) the seven displacements $u_{1C}, u_{2C}, u_{3C}, \theta_1, \theta_2, \theta_3, a_w$; (b) the six admissible strains $\varepsilon_C, \kappa_1, \kappa_2, \kappa_3, \alpha_w, \beta_w$; (c) the six active stresses $N, M_1, M_2, M_3, D_w, B_w$ and the two reactive shear-stresses T_2, T_3.

7.7 Unwarpable, cross-deformable, planar TWB

We consider a TWB, indifferently open or closed, whose cross-section is allowed to distort in its own plane, but is prevented from warping out-of-plane. The cross-section is symmetric with respect to the $\bar{\mathbf{a}}_2$-axis, which therefore contains the centroid G and the flexural center C. Loads are symmetric with respect to the $(\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2)$-plane, so that, while the beam is assumed to be unshearable, it extends and bends itself in this plane, undergoing a cross-section distortion which preserves the symmetry.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

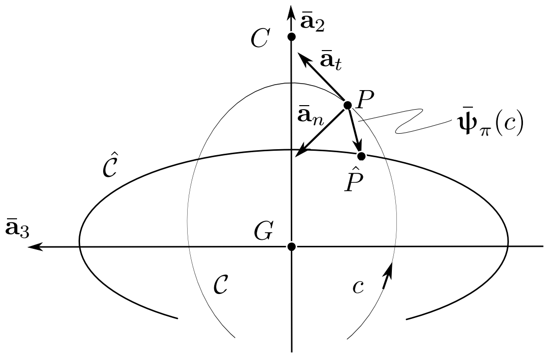


Figure 7.2: Mono-symmetric cross-section undergoing in-plane distortion.

7.7.1 Kinematics

Displacement field

From equations [7.33], [7.35] and [7.36], the position field of a TWB undergoing in-plane distortion only is:

$$\mathbf{x} = \mathbf{x}_C(s,t) + \mathbf{R}(s,t)(\bar{\mathbf{r}}(c) - \bar{\mathbf{r}}_C) + a_\pi(s,t)\,\mathbf{R}(s,t)\,\bar{\boldsymbol{\psi}}_\pi(c) \qquad [7.68]$$

where $\mathbf{R}(s,t)$ is a rotation of amplitude θ around the axis $\bar{\mathbf{a}}_3 \equiv \mathbf{a}_3$, $a_\pi(s,t)$ is an unknown amplitude function and $\bar{\boldsymbol{\psi}}_\pi(c)$ is a (sufficiently smooth) distortional vector, belonging to the cross-section plane π, which respects the symmetry with respect to the $\bar{\mathbf{a}}_2$-axis (Figure 7.2). The components of $\bar{\boldsymbol{\psi}}_\pi$ on the intrinsic basis to the middle-line $\mathcal{C}$, $(\bar{\mathbf{a}}_t(c), \bar{\mathbf{a}}_n(c))$, will be denoted by $\psi_t(c), \psi_n(c)$; the components in the Cartesian basis $(\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3)$ will be denoted by $\psi_2(c), \psi_3(c)$.

The displacement field is:

$$\mathbf{u} = \mathbf{u}_C(s,t) + (\mathbf{R}(s,t) - \mathbf{I})(\bar{\mathbf{r}}(c) - \bar{\mathbf{r}}_C) + a_\pi(s,t)\,\mathbf{R}(s,t)\,\bar{\boldsymbol{\psi}}_\pi \qquad [7.69]$$

which constitutes a *displacement-map* of the form [7.32a]. To enforce that $\bar{\boldsymbol{\psi}}_\pi$ is a pure distortion, we impose that it is orthogonal to a translation along $\bar{\mathbf{a}}_2$; therefore:

$$\int_{\mathcal{C}} \psi_2(c)\, b dc = 0 \qquad [7.70]$$

Longitudinal and transverse local strains

We adopt for the local strain vector $\mathbf{e}(s,c,t)$ the same expression we used for the beam as a whole. Since, due to the lack of warping, the local normals remain parallel

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

to the same direction $\mathbf{a}_1$, we have:

$$\mathbf{e} = \mathbf{R}^T(s,t)\,\mathbf{x}'(s,t) - \bar{\mathbf{a}}_1(s,c,t) \tag{7.71}$$

By using [7.68], the strain assumes the form:

$$\begin{aligned}\mathbf{e} &= \mathbf{R}^T\left[\mathbf{x}'_C + \mathbf{R}'(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C) + (\mathbf{R}'a_\pi + \mathbf{R}a'_\pi)\,\bar{\boldsymbol{\psi}}_\pi\right] - \bar{\mathbf{a}}_1 \\ &= \left(\mathbf{R}^T\mathbf{x}'_C - \bar{\mathbf{a}}_1\right) + \mathbf{R}^T\mathbf{R}'(\bar{\mathbf{r}} - \bar{\mathbf{r}}_C) + \left(\mathbf{R}^T\mathbf{R}'a_\pi + a'_\pi\right)\bar{\boldsymbol{\psi}}_\pi\end{aligned} \tag{7.72}$$

which can be *expressed in terms of the sole generalized strains of the 1D-model*, i.e.:

$$\mathbf{e} = \mathbf{e}_C + \mathbf{k} \times \left((\bar{\mathbf{r}} - \bar{\mathbf{r}}_C) + \alpha_\pi\bar{\boldsymbol{\psi}}_\pi\right) + \beta_\pi\bar{\boldsymbol{\psi}}_\pi \tag{7.73}$$

where the *warping strain* α_π and the *warping strain gradient* β_π have been introduced, defined by:

$$\alpha_\pi := a_\pi, \qquad \beta_\pi := a'_\pi \tag{7.74}$$

The local strain is the sum of three contributions: (a) the strain at the flexural axis; (b) the strain induced by the beam curvature, which accounts, as a second-order effect, for the in-plane distortion; and (c) a transverse-strain, induced by the variation of the distortion along the beam axis.

Since $\mathbf{k} =: \kappa\bar{\mathbf{a}}_3$, by decomposing $\mathbf{e}, \mathbf{e}_C$ along π and its normal (equation [7.45a,b]), we have:

$$\begin{aligned}&\varepsilon = \varepsilon_C + \kappa\bar{\mathbf{a}}_3 \times \left((\bar{\mathbf{r}} - \bar{\mathbf{r}}_C) + \alpha_\pi\bar{\boldsymbol{\psi}}_\pi\right)\cdot\bar{\mathbf{a}}_1 \\ &\boldsymbol{\gamma} = \boldsymbol{\gamma}_C + \beta_\pi\bar{\boldsymbol{\psi}}_\pi\end{aligned} \tag{7.75}$$

We assume $\boldsymbol{\gamma}_C = 0$ for unshearability, and neglect shear-strains normal to the middle-line. Therefore, by rearranging the first of them according to equation [7.46a], and projecting the second one on the tangent to $\mathcal{C}$, we obtain:

$$\begin{aligned}&\varepsilon = \varepsilon_G + \kappa\bar{\mathbf{a}}_3 \times \left(\bar{\mathbf{r}} + \alpha_\pi\bar{\boldsymbol{\psi}}_\pi\right)\cdot\bar{\mathbf{a}}_1 \\ &\gamma_t = \beta_\pi\psi_t\end{aligned} \tag{7.76}$$

which is a strain-map of the form [7.32b].

Distortional strains

The rigid motion of the cross-section leaves the ribs undeformed. The planar distortion of the cross-section, however, induces a *local torsion* $\kappa_t(c)$ *of the longitudinal fibers* (depending on the directrix-abscissa c), although the global

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

torsion is zero. Moreover, it triggers an *elongation* $\varepsilon_c(c)$ *and a flexure* $\chi_c(c)$ of the ribs[21]. To evaluate all these contribution, we linearize the kinematics, based on the fact that a_π is considered smaller of the rigid displacements of the sections (typically, the former is of the order of the thickness b, while the latter is of the order of the average diameter). Moreover, we assume that the ribs are shear-undeformable, and exploit the results we achieved concerning the planar arch (section 3.3). Accordingly, we have[22]:

$$\begin{aligned}\kappa_t &= \beta_\pi\left(\psi_n' + \bar{\kappa}\psi_t\right)\\ \chi_c &= \alpha_\pi\left(\psi_n' + \bar{\kappa}\psi_t\right)'\\ \varepsilon_c &= \alpha_\pi\left(\psi_t' - \bar{\kappa}\psi_n\right)\end{aligned} \qquad [7.77]$$

where $\bar{\kappa} = \bar{\kappa}(c)$ is the local curvature of the middle-line $\mathcal{C}$, and dashes denote differentiation with respect to the relevant spatial variable (namely c, when they are applied to ψ).

REMARK 7.4. In this model, the longitudinal fibers do not undergo flexure due to distortional displacements, since their sections have been assumed to remain parallel to the plane π in the current configuration. In contrast, distortion induces shear, already accounted by equation [7.75b].

Velocity and strain-rate field

The velocity field is supplied by time-differentiation of equation [7.68]; since:

$$\left(a_\pi(s,t)\,\mathbf{R}(s,t)\right)^{\cdot}\bar{\boldsymbol{\psi}}_\pi(c) = \left(\dot{a}_\pi\mathbf{R} + a_\pi\dot{\mathbf{R}}\right)\bar{\boldsymbol{\psi}}_\pi = \left(\dot{a}_\pi + a_\pi\dot{\mathbf{R}}\mathbf{R}^T\right)\boldsymbol{\psi}_\pi \qquad [7.78]$$

in which $\boldsymbol{\psi}_\pi := \mathbf{R}\bar{\boldsymbol{\psi}}_\pi$, we finally have:

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_C(s,t) + \boldsymbol{\omega}(s,t)\times\left(\mathbf{r}(c) - \mathbf{r}_C\right)\\ &\quad + \dot{a}_\pi(s,t)\,\boldsymbol{\psi}_\pi(c) + a_\pi(s,t)\,\boldsymbol{\omega}(s,t)\times\boldsymbol{\psi}_\pi(c)\end{aligned} \qquad [7.79]$$

in which $\boldsymbol{\omega} = \omega\mathbf{a}_3$ and the $\dot{a}_\pi$ is the *distortional velocity*. This is a velocity-map of the form [7.32c].

21. Here and in what follows, the index c denotes "along the abscissa c", or "circumferential".

22. We remember that the rotation of the centerline of the unshearable arch is $\theta_c = u_n' + \bar{\kappa}u_t$, with u_n, u_t the normal and tangential displacement components. Therefore, the torsion of the fibers is equal to $\partial_s\theta_c$, while the flexure of the ribs is $\partial_c\theta_c$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The strain-rate is obtained by time-differentiating the strains [7.76] and [7.77]:

$$\begin{aligned}
&\dot{\varepsilon} = \dot{\varepsilon}_G + \dot{\kappa}\bar{\mathbf{a}}_3 \times \left(\bar{\mathbf{r}} + \alpha_\pi \bar{\boldsymbol{\psi}}_\pi\right) \cdot \bar{\mathbf{a}}_1 + \kappa\bar{\mathbf{a}}_3 \times \dot{\alpha}_\pi \bar{\boldsymbol{\psi}}_\pi \cdot \bar{\mathbf{a}}_1 \\
&\dot{\gamma}_t = \dot{\beta}_\pi \psi_t \\
&\dot{\kappa}_t = \dot{\beta}_\pi \left(\psi_n' + \bar{\kappa}\psi_t\right) \\
&\dot{\chi}_c = \dot{\alpha}_\pi \left(\psi_n' + \bar{\kappa}\psi_t\right)' \\
&\dot{\varepsilon}_c = \dot{\alpha}_\pi \left(\psi_t' - \bar{\kappa}\psi_n\right)
\end{aligned} \qquad [7.80]$$

A strain-rate-map of the form [7.32d] is therefore obtained.

7.7.2 Identification procedure

We identify external forces, generalized stresses and elastic constants, in a way similar to that followed for the warpable beam. Symbols keep the same meaning as in the former case.

Generalized external force identification

The external virtual power for the 3D-model, using the velocity-map [7.79c], reads:

$$\begin{aligned}
\mathcal{P}_{ext} &= \int_{\mathcal{V}} \mathbf{b} \cdot \mathbf{v} dV + \sum_{H=A}^{B} \int_{\mathcal{A}_H} \mathbf{f}_H \cdot \mathbf{v}_H dA \\
&= \int_{\mathcal{S}} ds \int_{\mathcal{C}} \mathbf{b} \cdot \left(\mathbf{v}_C + \omega \mathbf{a}_3 \times (\mathbf{r} - \mathbf{r}_C) + \dot{a}_\pi \boldsymbol{\psi}_\pi + a_\pi \omega \mathbf{a}_3 \times \boldsymbol{\psi}_\pi\right) bdc \\
&+ \sum_{H=A}^{B} \int_{\mathcal{C}_H} \mathbf{f}_H \cdot (\mathbf{v}_{CH} + \omega_H \mathbf{a}_3 \times (\mathbf{r} - \mathbf{r}_C) \\
&+ \dot{a}_{\pi H} \boldsymbol{\psi}_\pi + a_{\pi H} \omega_H \mathbf{a}_3 \times \boldsymbol{\psi}_\pi) bdc
\end{aligned} \qquad [7.81]$$

Since the external virtual power [7.22] for the 1D-model reads:

$$\begin{aligned}
\mathcal{P}_{ext} &:= \int_{\mathcal{C}} \left(\mathbf{p} \cdot \mathbf{v}_C + c\omega + q_\pi \dot{a}_\pi\right) ds \\
&+ \sum_{H=A}^{B} \left(\mathbf{P}_H \cdot \mathbf{v}_{CH} + C_H \omega_H + Q_{\pi H} \dot{a}_{\pi H}\right)
\end{aligned} \qquad [7.82]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

the following generalized forces are identified:

$$\begin{aligned} \mathbf{p} &:= \int_{\mathcal{C}} \mathbf{b} b dc, & c &:= \int_{\mathcal{C}} (\mathbf{r} - \mathbf{r}_C + a_\pi \boldsymbol{\psi}_\pi) \times \mathbf{b} \cdot \mathbf{a}_3 b dc \\ \mathbf{P}_H &:= \int_{\mathcal{C}_H} \mathbf{f}_H b dc, & C_H &:= \int_{\mathcal{C}_H} (\mathbf{r} - \mathbf{r}_C + a_\pi \boldsymbol{\psi}_\pi) \times \mathbf{f}_H \cdot \mathbf{a}_3 b dc \\ q_w &:= \int_{\mathcal{C}} \boldsymbol{\psi}_\pi \cdot \mathbf{b} b dc, & Q_{wH} &:= \int_{\mathcal{C}_H} \boldsymbol{\psi}_\pi \cdot \mathbf{f}_H b dc \end{aligned} \qquad [7.83]$$

The generalized forces $\mathbf{p}, \mathbf{P}_H$, as well the couples c, C_H, are the resultant forces and moments applied to the cross-section; the latter, however, account for the change of the planar geometry. The distortional forces q_π and $Q_{\pi H}$ are the scalar product of the body- and surface-forces by the planar distortional mode.

REMARK 7.5. External forces orthogonal to the symmetry-plane, supposedly symmetric with respect to this plane, also contribute to the distortional forces. In contrast, they have no effect on the generalized forces of the locally rigid model, since they cancel by themselves.

Generalized stress identification

We write the internal virtual power for the 3D-model. By denoting by $\boldsymbol{\sigma} = \sigma \mathbf{a}_1 + \tau_t \mathbf{a}_t$ the stresses acting on the fibers, by m_t the torsional moment of the fibers, by m_c the flexural moment of the ribs and by σ_c the normal stress of the ribs, we have:

$$\begin{aligned} \mathcal{P}_{int} &= \int_{\mathcal{V}} \left(\sigma \dot{\varepsilon} + \tau_t \dot{\gamma}_t + m_t \dot{\kappa}_t + m_c \dot{\chi}_c + \sigma_c \dot{\varepsilon}_c\right) dV \\ &= \int_{\mathcal{S}} ds \int_{\mathcal{C}} \Big[\sigma(\dot{\varepsilon}_G + \dot{\kappa}\left(\mathbf{r} + \alpha_\pi \boldsymbol{\psi}_\pi\right) \times \mathbf{a}_1 \cdot \mathbf{a}_3 \\ &- \kappa \dot{\alpha}_\pi \mathbf{a}_2 \cdot \boldsymbol{\psi}_\pi) + \tau_t \dot{\beta}_\pi \psi_t\Big] b dc \\ &+ \int_{\mathcal{S}} ds \int_{\mathcal{C}} \Big[\dot{\beta}_\pi m_t \left(\psi_n' + \bar{\kappa}\psi_t\right) + \dot{\alpha}_\pi m_c \left(\psi_n' + \bar{\kappa}\psi_t\right)' \\ &+ \dot{\alpha}_\pi \sigma_c \left(\psi_t' - \bar{\kappa}\psi_n\right)\Big] b dc \end{aligned} \qquad [7.84]$$

where the strain-rate-map [7.80b] has been used and all vectors in the mixed products have been rotated. In addition, the internal virtual power [7.23] for the 1D-model,

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

accounting for unshearability, is:

$$\mathcal{P}_{int} := \int_{\mathcal{S}} \left(N\dot{\varepsilon}_G + M\dot{\kappa} + D_\pi \dot{\alpha}_\pi + B_\pi \dot{\beta}_\pi \right) ds \qquad [7.85]$$

where $\mathbf{m} =: M\mathbf{a}_3$ is the flexural moment. By equating the two expressions for the internal power, the generalized stresses are identified as:

$$\begin{aligned} N &= \int_{\mathcal{C}} \sigma b dc \\ M &= \int_{\mathcal{C}} \left[(\mathbf{r} + \alpha_\pi \boldsymbol{\psi}_\pi) \times \sigma \mathbf{a}_1 \cdot \mathbf{a}_3 \right] bdc \\ D_\pi &= \int_{\mathcal{C}} \left(m_c \left(\psi_n' + \bar{\kappa}\psi_t \right)' + \sigma_c \left(\psi_t' - \bar{\kappa}\psi_n \right) - \sigma\kappa \boldsymbol{\psi}_\pi \cdot \mathbf{a}_2 \right) bdc \\ B_\pi &= \int_{\mathcal{C}} \left[\tau_t \psi_t + m_t \left(\psi_n' + \bar{\kappa}\psi_t \right) \right] bdc \end{aligned} \qquad [7.86]$$

The first two generalized stresses can be interpreted as the resultant of the local stresses; however, in the moment evaluation, since the cross-section is deformed in its plane, the "arms" of the normal stresses are modified with respect to the reference configuration. The distortional stresses, as usual, represent the virtual power of the local stresses in the local distortional strain-rates.

REMARK 7.6. The variation of the planar geometry of the section was the argument exploited by Brazier [BRA 27] to explain why the "effective moment of inertia" of the cross-section is less than that of the undeformed cross-section.

Elastic constant identification

The elastic potential for the 3D-model reads:

$$\begin{aligned} U &:= \int_{\mathcal{S}} ds \int_{\mathcal{C}} \frac{1}{2} \left(E\varepsilon^2 + G\gamma_t^2 + \frac{1}{3} G b^2 \kappa_t^2 + \frac{1}{12} E b^2 \chi_c^2 + E\varepsilon_c^2 \right) bdc \\ &=: \int_{\mathcal{C}} \left(\phi_\varepsilon + \phi_\gamma + \phi_{\kappa_t} + \phi_{\chi_c} + \phi_{\varepsilon_c} \right) ds \end{aligned} \qquad [7.87]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By using strains [7.76], [7.75b] and [7.77], and exploiting the geometrical properties of the principal centroid axis and the orthogonality condition [7.70], we have:

$$\begin{aligned}
\phi_\varepsilon &= \frac{1}{2}E\int_{\mathcal{C}} \left[\varepsilon_G - (r_2+\alpha_\pi\psi_2)\,\kappa\right]^2 bdc \\
&= \frac{1}{2}EA\varepsilon_G^2 + \frac{1}{2}\left(EJ_3 + 2EJ_{3\psi}\alpha_\pi + EJ_\psi\alpha_\pi^2\right)\kappa^2 \\
\phi_\gamma &= \frac{1}{2}G\int_{\mathcal{C}} \beta_\pi^2\psi_t^2 bdc = \frac{1}{2}GJ_\gamma\beta_\pi^2 \\
\phi_{\kappa_t} &= \frac{1}{2}G\int_{\mathcal{C}} \frac{1}{3}b^3\beta_\pi^2\left(\psi_n' + \bar{\kappa}\psi_t\right)^2 dc = \frac{1}{2}GJ_{k_t}\beta_\pi^2 \\
\phi_{\chi_c} &= \frac{1}{2}E\int_{\mathcal{C}} \frac{1}{12}b^3\alpha_\pi^2\left(\psi_n' + \bar{\kappa}\psi_t\right)'^2 dc = \frac{1}{2}EA_\chi\alpha_\pi^2 \\
\phi_{\varepsilon_c} &= \frac{1}{2}E\int_{\mathcal{C}} b\alpha_\pi^2\left(\psi_t' - \bar{\kappa}\psi_n\right)^2 dc = \frac{1}{2}EA_\varepsilon\alpha_\pi^2
\end{aligned} \qquad [7.88]$$

where the following geometrical quantities have been introduced[23]:

$$\begin{aligned}
&J_3 = \int_{\mathcal{C}} r_2^2 bdc, \qquad J_{3\psi} = \int_{\mathcal{C}} r_2\psi_2 bdc, \qquad J_\psi = \int_{\mathcal{C}} \psi_2^2 bdc \\
&J_\gamma = \int_{\mathcal{C}} \psi_t^2 bdc, \qquad J_{k_t} = \frac{1}{3}\int_{\mathcal{C}} \left(\psi_n' + \bar{\kappa}\psi_t\right)^2 b^3 dc \\
&A_\chi = \frac{1}{12}\int_{\mathcal{C}} \left(\psi_n' + \bar{\kappa}\psi_t\right)'^2 b^3 dc, \qquad A_\varepsilon = \int_{\mathcal{C}} \left(\psi_t' - \bar{\kappa}\psi_n\right)^2 bdc
\end{aligned} \qquad [7.89]$$

23. In attributing a name to each quantity, we thought ψ_t, ψ_t to be lengths $[\mathrm{L}^1]$, according to the meaning they take in the Brazier theory (see section 8.5 ahead). As a result, $[A_\alpha] = \left[\mathrm{L}^2\right]$, $[J_\alpha] = \left[\mathrm{L}^4\right]$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with A the cross-section area. By requiring that $U = \frac{1}{2}\int_{\mathcal{C}} \phi ds$, we get the elastic potential for the 1D-model:

$$\begin{aligned}\phi &= \frac{1}{2}\left(EA\varepsilon_G^2 + EJ_3\kappa^2\right)\\ &+ \frac{1}{2}\left(EA_\chi + EA_\varepsilon\right)\alpha_\pi^2 + \frac{1}{2}\left(GJ_{k_t} + GJ_\gamma\right)\beta_\pi^2 \\ &+ EJ_{3\psi}\alpha_\pi\kappa^2 + \frac{1}{2}EJ_\psi\alpha_\pi^2\kappa^2\end{aligned} \quad [7.90]$$

from which the constitutive law follows by differentiation:

$$\begin{pmatrix} N \\ M \\ D_\pi \\ B_\pi \end{pmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & EJ_3 & 0 & 0 \\ 0 & 0 & EA_\chi + EA_\varepsilon & 0 \\ 0 & 0 & 0 & GJ_{k_t} + GJ_\gamma \end{bmatrix} \begin{pmatrix} \varepsilon_G \\ \kappa \\ \alpha_\pi \\ \beta_\pi \end{pmatrix} + \boldsymbol{f}(\boldsymbol{\varepsilon}) \quad [7.91]$$

where:

$$\boldsymbol{f}(\boldsymbol{\varepsilon}) := \begin{pmatrix} 0 \\ 2EJ_{3\psi}\kappa\alpha_\pi + EJ_\psi\kappa\alpha_\pi^2 \\ EJ_{3\psi}\kappa^2 + EJ_\psi\kappa^2\alpha_\pi \\ 0 \end{pmatrix} \quad [7.92]$$

It appears that coupling is only of nonlinear nature. If we assume small distortions, we can neglect terms proportional to EJ_ψ.

7.7.3 The Fundamental Problem

The Fundamental Problem for the 1D, planar, unshearable, unwarpable and cross-deformable TWB is governed by the following equations, when local distortion is described by just one configuration variable a_π:

1) The strain–displacement relationships [7.16a,f], [7.14c,d], specialized to the planar case:

$$\begin{aligned}\varepsilon_C &= (1 + u'_{1C})\cos\theta + u'_{2C}\sin\theta - 1\\ \kappa &= \theta'\\ \alpha_\pi &= a_\pi, \qquad \beta_\pi = a'_\pi\end{aligned} \quad [7.93]$$

2) The unshearability condition, derived from equation [7.16b]:

$$-(1 + u'_{1C})\sin\theta + u'_{2C}\cos\theta = 0 \quad [7.94]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

3) The balance equations [7.26], specialized to the planar case:

$$\begin{aligned} & N' - \kappa T + p_1 = 0 \\ & T' + \kappa N + p_2 = 0 \\ & M' + r_{2C} N' + (1 + \varepsilon_C) T + c = 0 \\ & B'_\pi - D_\pi + q_\pi = 0 \end{aligned} \qquad [7.95]$$

4) The elastic law [7.91], linking the active stresses to the admissible strains.

The mechanical boundary conditions follow from equations [7.29]:

$$\begin{aligned} & [(P_1 \pm N)\, v_{1C}]_H = 0, \qquad [(P_2 \pm T)\, v_{2C}]_H = 0 \\ & [(C \pm (M - r_{2C} N))\, \omega]_H = 0, \quad [(Q_\pi \pm B_\pi)\, \dot{a}_\pi]_H = 0, \quad H = A, B \end{aligned} \qquad [7.96]$$

The previous equations express the mixed formulation for the internally constrained TWB under study. They are a system of 13 equations in the following unknowns: (a) the four displacements $u_{1C}, u_{2C}, \theta, a_\pi$; (b) the four admissible strains $\varepsilon_C, \kappa, \alpha_\pi, \beta_\pi$; (c) the four active stresses N, M, D_π, B_π and the reactive shear-stress T.

7.8 Summary

In this chapter, we formulated 1D-models for TWB, whose cross-sections are susceptible to change their initial shape. We called these beams *locally deformable*, just referring to the cross-sections, as opposite to locally rigid beams, whose cross-sections behave as rigid bodies.

We first introduced a *direct model*, in the spirit of this book. The formulations start by considering a set of additional parameters with respect to those describing kinematics of locally rigid beams. They are scalar *distortional variables* $a_j\,(s,t)$, supposed to describe (in a rough way) the change of shape of the cross-section at the abscissa s and time t. These parameters were recognized to possess a double nature, namely, of displacements (since they describe the new configuration of the beam) and of strains (since they are identically zero in a rigid transformation of the beam). Accordingly, we used the symbol a_j to denote a displacement, and the symbol $\alpha_j := a_j$ to denote the distortional strain; moreover, the gradient of the distortion, $\beta_j := a'_j$ was recognized to be a further strain. Remarkably, kinematics of the rigid and distortional transformations was found to be uncoupled. The balance equations were derived by the VPP, after having introduced the dual stress quantities D_j, B_j, spending power on the strain-rates $\dot{\alpha}_j, \dot{\beta}_j$, respectively. They were named the *distortional stresses*. Again, the balance equations involving distortional stresses were found to be uncoupled from those relevant to locally rigid beams. The elastic

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

problem was completed by constitutive laws derived from an assumed elastic potential, which finally couples the two groups of equations.

The model so far described does not account for the (linear) properties of the flexural- (or shear-) center, which are instead useful to simplify the constitutive law. Therefore, a *two-axis beam* was considered, whose main axis is the locus of the flexural centers C, and the secondary axis that of the centroids G. The rigid kinematics of the cross-section was described with reference to C, so that displacement, velocity, rotation and spin denote quantities relevant to this point. Consistently, C-strains $\mathbf{e}_C, \mathbf{k}$ were derived, and their relationships with the more familiar G-strains $\mathbf{e}_G, \mathbf{k}$ were found. However, it was observed that longitudinal strains ε_G are more convenient than ε_C, while transverse strains γ_C are more convenient than γ_G, so that a *mixed description* was adopted. When the internal virtual power is written consistently, the axial force N assumes the meaning of internal force applied to the centroid, while the shear-force $\mathbf{t}_\pi$ assumes that of force applied to the flexural-center; consequently, $\mathbf{m}_\pi$ is the flexural moment with respect to the centroid, and M_1 is the torsional moment with respect to the flexural-center. The balance equations were derived by the VPP. Since velocities are referred to C, they assume the meaning of cardinal equations evaluated with respect to the same point C. Therefore, the moment $\mathbf{m}_\pi$ appears in the balance equations modified by an extra-term which is the torque of N (applied at G) with respect to C.

The direct approach was commented to be an elegant way to derive a 1D-model. However, it leaves the meaning of the distortional parameters, as well as that of the distortional stresses. Most importantly, the procedure leaves the question of how to choose the constants appearing in the constitutive law open. For these reasons, it has been found to be convenient to establish a procedure of identification from an (approximated) 3D-model of TWB. The identification is based on the construction of maps, relating the local quantities of the 3D-model to the global quantities of the 1D-model. Thus, a *displacement-map* relates the local displacements $\mathbf{u}$ to the global $\mathbf{u}_C, \mathbf{R}, a_j$; a *strain-map* links the local strains $\boldsymbol{\varepsilon}$ to the global strains $\mathbf{e}, \mathbf{k}, \alpha_j, \beta_j$; time-differentiation of the former relations provides a *velocity-map* and a *strain-rate-map*. Identification of the external (internal) forces is performed by equating the external (internal) virtual power for the two models, and using the velocity- and the strain-rate-maps, respectively. The constitutive law is a consequence of equating the elastic potentials of the two models and of the use of the strain-map.

The 3D-model employed for the identification is based on the usual hypotheses for TWB, justified by the smallness of the thickness. Moreover, it exploits the idea of the modern GBT, which describes the distortion of the cross-section as a linear combination of known shape-functions and unknown amplitude-functions $a_j(s,t)$. When, however, a strain measure must be introduced, it is easy to ascertain that not all strains are consistent with those adopted for the 1D-model, since they usually involve

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

quantities which do not appear in the 1D-model. Here, therefore, a *fiber-model* was used, made up of rods, each obeying the kinematic laws we established for the beam as a whole, which revealed itself to be well suited to the scope.

Two applications of the identification procedure were developed, concerning comparatively simple problems: (a) TWB embedded in a 3D-space, undergoing warping only, of amplitude a_w; and (b) TWB embedded in a 2D-space, undergoing cross-section planar distortions only, of amplitude a_π. To limit the algebra, we described the distortion by a unique a-parameter. For both problems, we identified the 1D external and the internal forces starting from body- and surface-forces acting on the 3D-model. In this way, we were able to give meaning to the distortional-forces, which are particular to locally deformable beams. Moreover, we observed that distortion enters in the expressions of generalized forces: for example, warping causes the tangential stresses to contribute to the flexural moment; analogously, the in-plane distortions modify the cross-section geometry, and therefore affect the flexural moment. Finally, we obtained nonlinear constitutive equations for hyperelastic TWB, revealing the existence of several couplings among the generalized strains.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Chapter 8

Distortion-Constrained Thin-Walled Beams

In this chapter, we consider locally deformable thin-walled beams (TWBs), whose distortional descriptors are not free, but are requested to satisfy additional conditions. Three different constraints are addressed, namely: (a) the Vlasov constraint for open TWB; (b) the Bredt constraint, for closed TWB; (c) the Brazier constraint, for planar TWB under flexure. The non-uniform torsion case is first discussed for educational purposes, aimed to put the elasto-reactive nature of the stresses in light. To this end, the mixed formulation is mainly followed, and a brief outline of the displacement formulation is given. The general problem is finally addressed, in the spirit of the displacement formulation, in which the distortional variables are taken as slave of the master locally rigid displacements.

8.1 Introduction

In Chapter 7, in dealing with models of locally deformable TWB, we introduced a set of kinematic descriptors $a_i\,(s,t)$, called *the distortional variables*, having the meaning of space- and time-dependent amplitudes of a set of distortional modes, assigned along the cross-section profile. These amplitudes were dealt with as *free variables*, like the displacements describing the rigid kinematics. The formulation led us to strain–displacement relationships [7.14c,d] and balance equation(s) [7.26c] involving distortional strains and stresses, respectively, uncoupled from homologous quantities for locally rigid models, equations [7.14a,b] and [7.26a,b]. Coupling, as we observed, is therefore only due to the constitutive law, which we identified via a

Mathematical Models of Beams and Cables Angelo Luongo and Daniele Zulli
© 2013 ISTE Ltd. Published 2013 by ISTE Ltd.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

fiber-model of three-dimensional (3D)-beam. On the other hand, in the qualitative discussion in section 7.1, we explained how the distortions of a TWB are mainly related to the rigid cross-section kinematics, namely the warping to the torsional curvature, and the in-plane distortions to the flexural curvature. Therefore, we would explore here the possibility to formulate internally constrained models of TWB, in which the distortional variables are rendered as slave of the locally-rigid displacements. We will call these models *distortion-constrained* TWBs.

In this chapter, we will show how to introduce the constraints and discuss the implications they have on the active/reactive nature of the stresses. The discussion is aimed: (a) to regain classical results from the linear theory, by clarifying the role of the internal constraints; (b) to extend the classical theories to the nonlinear range; (c) to give hints on how to chose the distortional modes. Once again, however, we will consider only one distortional variable to limit the extension of the formulas.

8.2 Internal constraints

We consider the Vlasov and Bredt constraints, able to relate the warping distortion α_w to the torsion κ_1 of open or closed TWB, respectively, whose cross-sections are susceptible to warp, but not to deform in their plane. On the grounds of kinematic or equilibrium considerations concerning the 3D-model, we will derive a kinematic constraint condition for the one-dimensional (1D)-model, namely:

$$\alpha_w = \kappa_1 \quad [8.1]$$

Then, we will consider a (Brazier) constraint, capable of relating the in-plane distortional strain α_π to the flexure $\kappa := \kappa_3$ of planar unwarpable TWB. Based on the equilibrium of the 3D-TWB, we will show that:

$$\alpha_\pi = \kappa^2 \quad [8.2]$$

8.2.1 The Vlasov constraint for open TWB

We consider a warpable, cross-undeformable 3D-TWB, having an *open* profile $\mathcal{C}$, and introduce a kinematic hypothesis by Vlasov [VLA 61], namely *the tangential shear-strain identically vanishes along the middle-line* $\mathcal{C}$, i.e. $\gamma_t(c) = 0 \; \forall c$ [1]. From a geometrical point of view, this hypothesis implies that the cross-section warps in such a way that the local normal $\mathbf{a}_w$ remains aligned with the current tangent $\mathbf{x}'$ to

1. The hypothesis is suggested by the fact that, in de Saint-Venant Problem of torsion, the shear-strains vary on the chord with a linear law by vanishing at the middle-line.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

the longitudinal fibers, after the latter incline themselves as a consequence of torsion. Concerning our model of a bundle of rods, not only the fiber passing through the flexural center is shear-undeformable, but *all* the rods are shear-undeformable. However, we will not satisfy this *nonlinear* kinematic constraint, but we will linearize it in the strains, by somewhat relaxing it. By remembering equation [7.49b], we enforce:

$$\kappa_1 (s,t)\, r_{nC} (c) + \alpha_w (s,t) \frac{d\psi_w (c)}{dc} + \text{h.o.t.} = 0 \qquad [8.3]$$

where $r_{nC} := (\bar{\mathbf{r}}(c) - \bar{\mathbf{r}}_C) \times \bar{\mathbf{a}}_t \cdot \bar{\mathbf{a}}_1$ and h.o.t. denotes quadratic term that we neglect. In order the previous equation holds for any s, t, equation [8.1] follows. Moreover, in order it holds for any c, then $d\psi_w/dc = -r_{nC}$, or:

$$d\psi_w = -2d\Omega_C \qquad [8.4]$$

Here $d\Omega_C := (1/2)\, r_{nC} dc$ is the (oriented) *elementary sectorial area* spanned by the radius CP, when its free-end P travels the line $\mathcal{C}$ by moving of an amount $\bar{\mathbf{a}}_t (c)\, dc$ (see Figure 8.1, where $r_{nC} > 0$, $d\Omega_C > 0$).

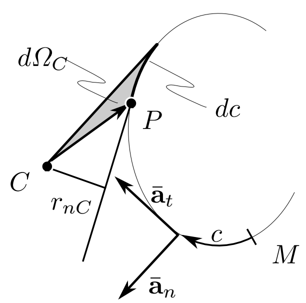


Figure 8.1: Sectorial area.

By integrating equation [8.4] between the origin M of the c-abscissa, and the generic point P, and enforcing the orthogonality condition [7.39a], it follows[2]:

$$\psi_w (c) = -2 \left(\Omega_C (c) - \bar{\Omega}_C \right) \qquad [8.5]$$

2. According to equation [8.5], the warping mode ψ_w has the dimension of an area, as we anticipated in Chapter 7. Consequently, a_w has the dimension of the inverse of a length, according to equation [8.1].

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where:

$$\Omega_C(c) := \int_M^P r_{nC}\, dc \qquad [8.6]$$

is the sectorial area referred to the flexural center, and $\bar{\Omega}_C := \left(\int_{\mathcal{C}} \Omega_C(c)\, b(c)\, dc\right)/A$ is its average value of $\Omega_C(c)$ on the area A, with $b(c)$ the thickness[3].

With the warping function satisfying $d\psi_w/dc = -r_{nC}$, the following relationships hold among some of the cross-section geometric characteristics we found for the unconstrained beam [7.62]:

$$J_n = J_w = -J_{nw} \qquad [8.7]$$

REMARK 8.1. The Vlasov constraint not only links the warping amplitude to the torsion (equation [8.1]), but it even leads us to choose a specific distortional mode (equation [8.5]).

REMARK 8.2. The previous results are well-known in the *linear* theory of open TWB. However, they also hold in the *nonlinear* field, since the torsion is allowed to be large here. On the other hand, we neglected the flexural–torsional coupling in the constraint equation. To account for this effect too, a unique distortional mode is not sufficient.

8.2.2 The Bredt constraint for tubular TWB

Now we consider a warpable, cross-undeformable, TWB having a *closed* profile $\mathcal{C}$. For the sake of simplicity, we will confine ourselves to a *tubular* cross-section, i.e. degree-2 internally connected profiles. The extension to higher degree connections (cellular cross-sections), although not difficult, is omitted here, and left to the reader[4].

According to Bredt's approximate theory of torsion, we introduce the following kinematic hypothesis: *the flow of tangential shear-strain across the chord Ξ is*

3. It is easy to check that the remaining two orthogonality conditions [7.39b,c] lead to determine the coordinates of the flexural center:

$$r_{2C} := \frac{2}{J_2}\int_{\mathcal{C}} r_3(c)\,\Omega_G(c)\,b(c)\,dc, \qquad r_{3C} := -\frac{2}{J_3}\int_{\mathcal{C}} r_2(c)\,\Omega_G(c)\,b(c)\,dc$$

As a matter of fact, from equation [8.6], $\Omega_C = \Omega_G - \bar{\mathbf{r}}_C \times (\bar{\mathbf{r}} - \bar{\mathbf{r}}_M)\cdot \bar{\mathbf{a}}_1$ follows. By enforcing the orthogonality conditions [7.39], accounting for the properties of principal axes, and solving with respect to the components of $\bar{\mathbf{r}}_C$, the previous equations are obtained.

4. For cellular sections, the Bredt condition must be enforced for each branch, and the monodromy condition enforced for each independent closed path.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

constant along the middle-line $\mathcal{C}$ and proportional to the torsion, i.e. $\gamma_t(s,c,t)\,b(c) = \kappa_1(s,t)\mathcal{Q}$, with $\mathcal{Q} = \text{constant}$[5]. To within a second-order error, we already discussed, the constraint becomes:

$$\kappa_1(s,t)\,r_{nC}(c) + \alpha_w(s,t)\,\frac{d\psi_w(c)}{dc} + \text{h.o.t.} = \kappa_1(s,t)\frac{\mathcal{Q}}{b(c)} \qquad [8.8]$$

From this, equation [8.1] is recovered, together with:

$$d\psi_w = \left(\frac{\mathcal{Q}}{b(c)} - r_{nC}\right) dc \qquad [8.9]$$

By integrating the latter:

$$\psi_w = \mathcal{Q}\int_M^P \frac{dc}{b(c)} - 2\varOmega_C(c) + \text{const} \qquad [8.10]$$

where the constant is determined by requiring that the average value of ψ_w vanishes. The constant $\mathcal{Q}$ is instead determined by the *monodromy condition*, which requires that the closed line integral of $d\psi_w$ vanishes, i.e. $\oint_{\mathcal{C}} d\psi_w = 0$[6]. From this, it follows:

$$\mathcal{Q} = \frac{2\varOmega}{\oint_{\mathcal{C}} \frac{dc}{b(c)}} \qquad [8.11]$$

where $\varOmega$ is the area of the region enclosed by the middle-line $\mathcal{C}$[7].

When the warping function satisfies equation [8.9], it follows that some of the geometrical characteristics [7.62], relevant to the unconstrained TWB, are *not* linearly independent, since:

$$J_n + 2J_{nw} + J_w = \frac{4\varOmega^2}{\oint_{\mathcal{C}} \frac{dc}{b(c)}} =: J_{BR} \qquad [8.12]$$

where J_{BR} *is the Bredt torsional stiffness of a tubular cross-section.*

5. The hypothesis is suggested by the fact that, in de Saint-Venant Problem of *uniform* torsion, the shear-strains (and stresses) are a solenoidal field (i.e. their in-plane divergence is zero everywhere), this entailing, via the Gauss theorem, the constancy of the flow.

6. The condition entails that $d\psi_w$ is an *exact differential*.

7. Do not confuse the area $\varOmega$ with the sectorial area $\varOmega_C(c)$. Of course $\varOmega = \varOmega_C(l_{\mathcal{C}}) - \varOmega_C(0)$, where $l_{\mathcal{C}}$ is the length of $\mathcal{C}$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

8.2.3 The Brazier constraint for planar TWB

We consider a TWB, with a symmetric cross-section, bent in the plane of symmetry, undergoing distortion with no-warping. Our goal is to link the amplitude of the distortion to the curvature. To this end, we will closely follow the reasoning by Brazier [BRA 27], who approximately evaluated the distortion: (a) by ignoring the effect of the distortion itself on the stress; (b) by ignoring the effect of the extension.

When the beam is bent in the $(\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2)$-plane, with a curvature $\kappa\bar{\mathbf{a}}_3$[8], the generic fiber belonging to the middle surface of the TWB, put at the distance $\bar{\mathbf{r}}(c)$ by G, is solicited by a normal stress[9]:

$$\sigma = E\kappa\bar{\mathbf{a}}_3 \times \bar{\mathbf{r}}(c) \cdot \bar{\mathbf{a}}_1 = -E\kappa r_2(c) \qquad [8.13]$$

Therefore, the elementary cylinder of cross-section area $b(c)\,dc$, centered on the fiber, is solicited by an axial force:

$$dN = \sigma(c)\,b(c)\,dc = -E\kappa r_2(c)\,b(c)\,dc \qquad [8.14]$$

Since the curvature of this cylinder is (approximately) equal to that of the centroidal fiber, $\kappa\bar{\mathbf{a}}_3$, the bent cylinder is in equilibrium if and only if an external force per unit length $f^\star(c)\,dc$, acting along $\bar{\mathbf{a}}_2$, is (temporarily) applied on it, with[10]:

$$f^\star(c)\,dc = -\kappa dN(c) = E\kappa^2 r_2(c)\,b(c)\,dc \qquad [8.15]$$

Since this force must be removed, as it has been artificially introduced, a force equal and opposite must be applied, namely:

$$f(c) = -f^\star(c) = -E\kappa^2(s,t)\,r_2(c)\,b(c) \qquad [8.16]$$

This latter is responsible for the cross-section distortion. Note that f linearly depends on the distance r_2, and its resultant is $\int_{\mathcal{C}} f(c)\,dc = 0$; therefore, distortion is produced by self-equilibrated forces.

To evaluate the distortion of the cross-section, we take an infinitesimal segment of TWB of unitary length $ds = 1$ and consider it as a planar frame in the $(\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3)$-plane,

8. We understood the dependence on s, t, by considering these quantities as fixed. In contrast, we highlight the dependence on the abscissa c of a given cross-section.

9. Note that, consistently with the equation [7.76], the stress should be written to $\sigma = E\left(\varepsilon_G + \kappa\bar{\mathbf{a}}_3 \times \left(\bar{\mathbf{r}} + \alpha_\pi\bar{\boldsymbol{\psi}}_\pi\right) \cdot \bar{\mathbf{a}}_1\right)$; therefore, $\varepsilon_G, \alpha_\pi$ must been neglected, according to Brazier.

10. Remember that the balance equation of a cable, governing the equilibrium along the local normal (here confused with $\bar{\mathbf{a}}_2$), becomes $p_n = -\kappa N$ (with extension neglected and symbols updated). By differentiating it with respect to c, and by letting $f^\star(c) := dp_n/dc$, equation [8.15] follows. Therefore $f^\star(c)$ is a force per unit of area.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

made of beams with rectangular cross-sections $ds \times b(c)$. Then, we load the frame by a surface force $\hat{f}(c) = -Er_2(c)\, b(c)$, corresponding to a unitary curvature $\kappa = 1$, and evaluate the *linear* static response of the frame, $\bar{\boldsymbol{\psi}}_\pi = \psi_2(c)\,\bar{\mathbf{a}}_2 + \psi_3(c)\,\bar{\mathbf{a}}_3$, which depends only on the geometrical characteristics of the cross-section. Hence, the response to $f = \kappa^2(s,t)\,\hat{f}(c)$ is $\mathbf{w} = \kappa^2(s,t)\,\bar{\boldsymbol{\psi}}_\pi(c)$. We can conclude that the distortional amplitude a_π (and therefore the distortional strain α_π) coincide with the squared curvature, i.e. equation [8.2] holds. We will call this relationship the *Brazier constraint.*

REMARK 8.3. Note that, differently from the Vlasov and Bredt constraints, which are linear, the Brazier constraint is nonlinear.

REMARK 8.4. As for Vlasov's and Bredt's, the Brazier constraint leads to the selection of a specific distorsional mode.

8.3 The non-uniform torsion problem for bi-symmetric cross-sections

Before addressing the general problem for flexural–flexural–torsional–extensional TWB undergoing distortion, it is instructive to analyze the particular case of pure torsion. This problem, indeed, explains in the simplest way the role played by the internal constraints and the reactive stresses. We start by formulating the unconstrained problem; then, we address the constrained problem, first by the mixed and then by the displacement formulations.

8.3.1 The unconstrained model

Let us consider a 1D-model of warpable and cross-undeformable TWB, exclusively loaded by torsional couples $\mathbf{c} = c\bar{\mathbf{a}}_1, \mathbf{C}_H = C_H\bar{\mathbf{a}}_1$, with warping forces q_w, Q_{wH} assumed to be zero for the sake of simplicity. The model is a particular case of the general 1D-model illustrated in section 7.3, here re-derived for clarity.

Twisted and shortened beams

First we observe that, due to nonlinearities, extension, flexure, torsion and warping are fully coupled; therefore, even if the beam is loaded by torsional couples only, it extends and bends itself, so that the problem is not simpler than the general problem. As a matter of fact, we check that, when the current configuration is rectilinear, i.e. $\kappa_1 \neq 0$, $\kappa_2 = \kappa_3 = 0$, the purely torsional state of stress $M_1 \neq 0$, $N = T_2 = T_3 = M_2 = M_3 = 0$ does satisfy the balance equations [7.28a,b,c,e,f] and the relevant

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

boundary conditions [7.29a,b,c,e,f]. However, this solution is *not* compatible with the constitutive law [7.66]. As an example, $EY_{2C}\kappa_1^2$ requires $M_2 \neq 0$. Therefore, with the aim to tackle the simplest possible problem, we limit ourselves to bi-symmetric cross-sections (e.g. the I-section), for which some elastic constants appearing in the constitutive law vanish. By accounting for definitions [7.62], and for the fact that ψ_w is antisymmetric on such cross-sections, we have:

$$Y_{2C} = Y_{3C} = 0, \qquad Y_{w2} = Y_{w3} = 0 \tag{8.17}$$

which imply $M_2 = M_3 = 0$, as we wanted. On the other hand, it follows from the same constitutive law that $N = 0$ entails *a non-zero axial shortening*, which always accompanies the twist, equal to:

$$\varepsilon_G = -\rho_C^2\kappa_1^2 \tag{8.18}$$

where $\rho_C^2 := J_C/A$. Therefore, the constitutive law for the twisted (and shortened) beam becomes:

$$\begin{pmatrix} M_1 \\ D_w \\ B_w \end{pmatrix} = \begin{bmatrix} GJ_{SV} + GJ_n & GJ_{nw} & 0 \\ GJ_{nw} & GJ_w & 0 \\ 0 & 0 & E\Gamma_w \end{bmatrix} \begin{pmatrix} \kappa_1 \\ \alpha_w \\ \beta_w \end{pmatrix} + \begin{pmatrix} E\Gamma_{wC}\beta_w\kappa_1 - EJ_C\rho_C^2\kappa_1^3 \\ 0 \\ \frac{1}{2}E\Gamma_{wC}\kappa_1^2 \end{pmatrix} \tag{8.19}$$

REMARK 8.5. Note that the shortening produces a softening-type cubic term in the torsional moment law. Indeed, in the fiber model, the part of the elastic energy necessary to extend the fibers is subtracted from that necessary to produce local shear strains.

The Fundamental Problem

Under these assumptions, the only non-zero displacements are the twist angle $\theta := \theta_1$ and the warping amplitude a_w; strains consist of the torsional curvature κ_1 and distortional quantities α_w, β_w. The strain–displacement relationships are:

$$\kappa_1 = \theta', \qquad \alpha_w = a_w, \qquad \beta_w = a_w \tag{8.20}$$

The strain-rates are:

$$\dot\kappa_1 = \dot\theta', \qquad \dot\alpha_w = \dot a_w, \qquad \dot\beta_w = \dot a_w' \tag{8.21}$$

and the spin is:

$$\omega = \dot\theta \tag{8.22}$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By denoting the dual stresses (torsional moment, distortional and bi-distortional stresses, respectively) by M_1, D_w, B_w, the Virtual Power Principle (VPP) becomes:

$$\int_{\mathcal{S}} \left(M_1 \dot{\kappa}_1 + D_w \dot{\alpha}_w + B_w \dot{\beta}_w \right) ds = \int_{\mathcal{S}} c\omega ds + \sum_{H=A}^{B} C_H \omega_H \quad [8.23]$$

or, by using equations [8.21] and [8.22]:

$$\int_{\mathcal{S}} \left(M_1 \dot{\theta}' + D_w \dot{a}_w + B_w \dot{a}_w' \right) ds = \int_{\mathcal{S}} c\dot{\theta} ds + \sum_{H=A}^{B} C_H \dot{\theta}_H, \quad \forall \left(\dot{\theta}, \dot{a}_w \right) \quad [8.24]$$

By performing an integration by parts, we get:

$$\begin{aligned} &\int_{\mathcal{S}} \left[(M_1' + c)\, \dot{\theta} + (B_w' - D_w)\, \dot{a}_w \right] ds \\ &+ \sum_{H=A}^{B} \left[(C_H \pm M_{1H})\, \dot{\theta}_H \pm B_{wH} \dot{a}_{wH} \right] = 0, \qquad \forall \left(\dot{\theta}, \dot{a}_w \right) \end{aligned} \quad [8.25]$$

from which the following balance equations are derived:

$$\begin{aligned} &M_1' + c = 0 \\ &B_w' - D_w = 0 \end{aligned} \quad [8.26]$$

together with the boundary conditions:

$$\begin{aligned} &(C_H \pm M_{1H})\, \dot{\theta}_H = 0 \\ &\pm B_{wH} \dot{a}_H = 0 \end{aligned} \quad [8.27]$$

The constitutive law is given by equation [8.19].

To summarize, the unconstrained nonlinear problem of torsion for bi-symmetric cross-sections is governed by eight equations [8.20] and [8.26], [8.19], plus boundary conditions, in the eight unknowns $(\theta, a_w; \kappa_1, \alpha_w, \beta_w; M_1, D_w, B_w)$.

8.3.2 The mixed formulation for the constrained model

We introduce the constraint $\kappa_1 = \alpha_w$ in the unconstrained model. We first perform this task according to the *mixed* formulation, which better highlights the role of the active and reactive stresses.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Active and reactive stresses

First we observe that this kind of constraint is slightly more general than those so far considered in this book, but it falls in the class of the "general linear constraints" we discussed in section 1.3.1. By following the approach illustrated there, we split the internal virtual power in an active and a reactive part, i.e.:

$$\mathcal{P}_{int} = \mathcal{P}_{act} + \mathcal{P}_{react} = \int_{\mathcal{S}} \left(M_{1a}\dot{\kappa}_1 + D_{wa}\dot{\alpha}_w + B_{wa}\dot{\beta}_w \right) ds \\ + \int_{\mathcal{S}} R\left(\dot{\alpha}_w - \dot{\kappa}_1\right) ds \qquad [8.28]$$

or:

$$\mathcal{P}_{int} = \int_{\mathcal{S}} \left[\left(M_{1a} - R\right)\dot{\kappa}_1 + \left(D_{wa} + R\right)\dot{\alpha}_w + B_{wa}\dot{\beta}_w \right] ds \qquad [8.29]$$

where the index a denotes the active component of the stress, and R is a Lagrangian multiplier affecting the strain-rate constraint $\dot{\alpha}_w - \dot{\kappa}_1 = 0$. Note that R spends zero power in all the kinematically admissible virtual motions, according to the "Perfect Constraint Postulate"[11]. This expression shows that the total stresses that conjugate to the strains are:

$$M_1 := M_{1a} - R \\ D_w := D_{wa} + R \qquad [8.30] \\ B_w := B_{wa}$$

Thus, although the bi-distortional stress B_w is purely active, the torsional moment M_1 and the distortional stress D_w are of a mixed active-reactive nature, but their reactive parts are related.

The balance equations

To obtain the balance equations, we write the VPP [8.23], but with the Lagrangian multiplier incorporated, and use equations [8.21] and [8.22], i.e.:

$$\int_{\mathcal{S}} \left[\left(M_{1a} - R\right)\dot{\theta}' + \left(D_{wa} + R\right)\dot{a}_w + B_w\dot{a}_w' \right] ds = \int_{\mathcal{S}} c\dot{\theta} ds + \sum_{H=A}^{B} C_H\dot{\theta}_H \\ \forall \left(\dot{\theta}, \dot{a}_w\right) \qquad [8.31]$$

11. It could be useful to think to the reactive virtual power as the power that the reactive stresses would spend in the strain-rates *if they were allowed to violate* the constraint condition.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Since this coincides with equation [8.24], if equations [8.30a,b] are taken into account, the same balance equations [8.26] and [8.27] follow, but with the stresses split, i.e.:

$$\begin{aligned}&(M_{1a} - R)' + c = 0\\&B_w' - (D_{wa} + R) = 0\end{aligned} \qquad [8.32]$$

and:

$$\begin{aligned}&[C \pm (M_{1a} - R)]_H \, \dot{\theta}_H = 0\\&B_{wH} \dot{a}_{wH} = 0\end{aligned} \qquad [8.33]$$

The constitutive law

To derive the constitutive law for the active part of the stress, we use equation [8.19], in which we rename M_1, D_w as M_{1a}, D_{wa}, and we introduce the constraint $\kappa_1 = \alpha_w$, thus obtaining:

$$\begin{pmatrix} M_{1a} \\ D_{wa} \\ B_w \end{pmatrix} = \begin{bmatrix} G\,(J_{SV} + J_n + J_{nw}) & 0 \\ G\,(J_w + J_{nw}) & 0 \\ 0 & E\varGamma_w \end{bmatrix} \begin{pmatrix} \kappa_1 \\ \beta_w \end{pmatrix} + \begin{pmatrix} E\varGamma_{wC}\beta_w\kappa_1 - EJ_C\rho_c^2\kappa_1^3 \\ 0 \\ \frac{1}{2}E\varGamma_{wC}\kappa_1^2 \end{pmatrix} \qquad [8.34]$$

The Fundamental Problem

By summarizing, the constrained nonlinear problem of torsion, according to the mixed formulation, is governed by nine equations [8.20], [8.1], [8.32] and [8.34], plus boundary conditions, in the nine unknowns $(\theta, a_w; \kappa_1, \alpha_w, \beta_w; M_{1a}, D_{wa}, B_w; R)$.

The condensed problem

Although the condensed equations could be more straightforwardly obtained by the displacement formulation, we want to derive them in the context of the mixed formulation, aimed to throw light on the question.

Since $\alpha_w = \kappa_1$, the strain–displacement relationships [8.20] can be expressed in terms of the twist only, i.e.:

$$\kappa_1 = \theta', \quad \beta_w = \theta'' \qquad [8.35]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

By eliminating the reactive stress between equations [8.32], we obtain a unique balance equation linking the three active stresses, namely:

$$(M_{1a} + D_{wa})' - B_w'' + c = 0 \quad [8.36]$$

together with the condensed boundary conditions:

$$\begin{aligned} &[C \pm (M_{1a} + D_{wa} - B_w')]_H \, \dot{\theta}_H = 0 \\ &B_{wH} \dot{a}_{wH} = 0 \end{aligned} \quad [8.37]$$

Note that, in these equations, only the sum $M_{1a} + D_{wa}$ appears, not each stress individually. Therefore, we can condense also the elastic law [8.34], by adding the first two rows to each other, thus obtaining:

$$\begin{aligned} \begin{pmatrix} M_{1a} + D_{wa} \\ B_w \end{pmatrix} &= \begin{bmatrix} G\,(J_{SV} + J_n + 2J_{nw} + J_w) & 0 \\ 0 & E\Gamma_w \end{bmatrix} \begin{pmatrix} \kappa_1 \\ \beta_w \end{pmatrix} \\ &\quad + \begin{pmatrix} E\Gamma_{wC}\beta_w\kappa_1 - EJ_C\rho_c^2\kappa_1^3 \\ \frac{1}{2}E\Gamma_{wC}\kappa_1^2 \end{pmatrix} \end{aligned} \quad [8.38]$$

These equations specialize as follows to *open* or *tubular* TWB, when the relationship among the elastic constants, equations [8.7] or [8.12], is accounted for:

1) *Open TWB*. Because of equations [8.7], $D_{wa} = 0$ follows from equation [8.34b], so that *the distortional stress $D_w = R$ is just of reactive type*; in contrast, the torsional moment $M_1 = M_{1a} - R$ is partially active and partially reactive[12]. The condensed elastic law [8.38] becomes:

$$\begin{pmatrix} M_{1a} \\ B_w \end{pmatrix} = \begin{bmatrix} GJ_{SV} & 0 \\ 0 & E\Gamma_w \end{bmatrix} \begin{pmatrix} \kappa_1 \\ \beta_w \end{pmatrix} + \begin{pmatrix} E\Gamma_{wC}\beta_w\kappa_1 - EJ_C\rho_c^2\kappa_1^3 \\ \frac{1}{2}E\Gamma_{wC}\kappa_1^2 \end{pmatrix} \quad [8.39]$$

2) *Tubular TWB*. Because of equations [8.12], it turns out that $D_{wa} \neq 0$, so that *both M_1 and D_w are made of an active and a reactive quota*. The condensed elastic law [8.38] becomes:

$$\begin{pmatrix} M_{1a} + D_{wa} \\ B_w \end{pmatrix} = \begin{bmatrix} GJ_{BR} & 0 \\ 0 & E\Gamma_w \end{bmatrix} \begin{pmatrix} \kappa_1 \\ \beta_w \end{pmatrix} + \begin{pmatrix} E\Gamma_{wC}\beta_w\kappa_1 - EJ_C\rho_c^2\kappa_1^3 \\ \frac{1}{2}E\Gamma_{wC}\kappa_1^2 \end{pmatrix} \quad [8.40]$$

in which, being $J_{SV} \ll J_{BR}$, the former has been neglected with respect to the latter.

In summary, the condensed problem consists of: two strain–displacement relationships [8.35], one balance equation [8.36], two elastic laws [8.39] or [8.40],

12. The reactive torsional moment R contributes to the equilibrium. It is occasionally named in literature the "secondary torsional moment", or the "bi-shear". The bi-distortional stress B_w is also known as "bi-moment".

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

plus boundary conditions, in the five unknowns $(\theta; \kappa_1, \beta_w; M_{1a} + D_{wa}; B_w)$. When these are combined in terms of the twist angle only, they lead to:

$$\begin{aligned} & GJ_t\theta'' - E\Gamma_w\theta'''' - 3EJ_C\rho_c^2\theta'^2\theta'' + c = 0 \\ & \left[C \pm \left(GJ_t\theta' - E\Gamma_w\theta''' - EJ_C\rho_c^2\theta'^3\right)\right]_H \dot{\theta}_H = 0 \\ & \left[E\Gamma_w\theta'' + \frac{1}{2}E\Gamma_{wC}\theta'^2\right]_H \dot{a}_{wH} = 0 \end{aligned} \quad [8.41]$$

where:

$$J_t := \begin{cases} GJ_{SV} & \text{for open TWB} \\ GJ_{BR} & \text{for tubular TWB} \end{cases} \quad [8.42]$$

If these equations are linearized in θ, the well-known equations of the Vlasov theory of the torsion are recovered.

REMARK 8.6. When, as a particular case, the torsion is uniform, then, from equations [8.35], $\kappa_1 = \theta' = \text{const}$ and $\beta_w = \theta'' = 0$. If we neglect the nonlinear terms, then $B_w = 0$ follows from equations [8.39b] or [8.40b]. This entails, from equation [8.32b], that $R = -D_a$, so that the reactive moment vanishes in open TWB, but it is different from zero in tubular TWB.

8.3.3 The displacement formulation for the constrained model

The equilibrium equations [8.41], expressed in terms of the twist angle only, can, of course, directly be obtained via the displacement approach. This way is shorter, although it hides the role of the reactive stresses. We will illustrate the method here, as an exercise, in view of applying this method ahead.

When $\alpha_w = \kappa_1$ is substituted in the internal virtual power (left hand member of equation [8.23]), this reduces to:

$$\mathcal{P}_{int} = \int_{\mathcal{S}} \left(M_t\dot{\kappa}_1 + B_w\dot{\beta}_w\right) ds \quad [8.43]$$

where:

$$M_t := M_1 + D_w \quad [8.44]$$

is the generalized active stress dual of the torsion-rate. According to the displacement method, we have to eliminate a slave variable from the constraint $\alpha_w = \kappa_1$, or, in terms of

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

displacements, from $a_w = \theta'$. By taking a_w as slave and θ as master variable, we obtain equations [8.35]. The VPP, equation [8.23], therefore changes into:

$$\int_{\mathcal{S}} \left(M_t \dot{\theta}' + B_w \dot{\theta}'' \right) ds = \int_{\mathcal{S}} c\dot{\theta} ds + \sum_{H=A}^{B} C_H \dot{\theta}_H, \qquad \forall \dot{\theta} \tag{8.45}$$

which, after two integrations by parts, becomes:

$$\begin{aligned} &\int_{\mathcal{S}} \left(M_t' - B_w'' + c \right) \dot{\theta} ds \\ &\quad + \sum_{H=A}^{B} \left[C_H \pm \left(M_t - B_w' \right) \right] \dot{\theta}_H \mp \sum_{H=A}^{B} B_{wH} \dot{\theta}'_H = 0, \qquad \forall \dot{\theta} \end{aligned} \tag{8.46}$$

The field equation follows:

$$M_t' - B_w'' + c = 0 \tag{8.47}$$

with the associated boundary condition:

$$\begin{aligned} &\left[C \pm \left(M_t - B_w' \right) \right]_H \dot{\theta}_H = 0 \\ &\mp B_{wH} \dot{\theta}'_H = 0 \end{aligned} \tag{8.48}$$

These condensed equilibrium equations are identical to [8.36] and [8.37], found via the mixed formulation, if we observe that $M_t = M_1 + D_w \equiv M_{1a} + D_{wa}$. With the same substitution, the constitutive law is still expressed by equation [8.38]. Hence, the same equations [8.41] follow.

8.4 The general problem for warpable TWB

Now, we come back to the general problem of warpable, distortion-constrained, TWB. In formulating the model, we will follow the displacement method in order to condense the reactive part of the torsional moment. However, we will keep the shear forces in the model, so that, strictly speaking, the model obeys the hybrid formulation.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The balance equations for the constrained problem

The internal virtual power [7.58], once the constraint $\alpha_w = \kappa_1$ (in addition to unshearability) has been accounted for, becomes:

$$\mathcal{P}_{int} := \int_{\mathcal{S}} \left(N\dot{\varepsilon}_G + M_t\dot{\kappa}_1 + \mathbf{m}_\pi \cdot \mathbf{R}\dot{\mathbf{k}}_\pi + B_w\dot{\beta}_w \right) ds \qquad [8.49]$$

where $M_t := M_1 + D_w$. Similarly, since $a_w = \kappa_1$ also follows from the constraint, the external virtual power [7.55] assumes the form:

$$\begin{aligned}\mathcal{P}_{ext} :=& \int_{\mathcal{S}} \left(\mathbf{p} \cdot \mathbf{v}_C + \mathbf{c} \cdot \boldsymbol{\omega} + q_w\dot{\kappa}_1 \right) ds \\ &+ \sum_{H=A}^{B} \left(\mathbf{P}_H \cdot \mathbf{v}_{CH} + \mathbf{C}_H \cdot \boldsymbol{\omega}_H + Q_{wH}\dot{\kappa}_{1H} \right)\end{aligned} \qquad [8.50]$$

The balance equations are supplied by the VPP, when one uses the previous expressions for the internal and external powers. According to the hybrid formulation, we add the zero-terms $\mathbf{t}_\pi \cdot \mathbf{R}\dot{\boldsymbol{\gamma}}_C$ to the equation, where $\mathbf{t}_\pi$ assumes the meaning of vector of Lagrangian parameters. Moreover, by accounting for equation [7.21a] in order to change the pole (as we did in section 7.3), the VPP becomes:

$$\begin{aligned}&\int_{\mathcal{S}} \big(N\dot{\varepsilon}_C + \mathbf{t}_\pi \cdot \mathbf{R}\dot{\boldsymbol{\gamma}}_C + (M_t - q_w)\,\dot{\kappa}_1 \\ &\quad + (\mathbf{m}_\pi - \mathbf{r}_C \times N\mathbf{a}_1) \cdot \mathbf{R}\dot{\mathbf{k}}_\pi + B_w\dot{\beta}_w \big) ds \\ &\quad = \int_{\mathcal{S}} \left(\mathbf{p} \cdot \mathbf{v}_C + \mathbf{c} \cdot \boldsymbol{\omega} \right) ds + \sum_{H=A}^{B} \left(\mathbf{P}_H \cdot \mathbf{v}_{CH} + \mathbf{C}_H \cdot \boldsymbol{\omega}_H + Q_{wH}\dot{\kappa}_{1H} \right)\end{aligned} \qquad [8.51]$$

Using the strain-rate constraints $\dot{\beta}_w = \dot{\alpha}'_w = \dot{\kappa}'_1$, and integrating by parts, it follows that:

$$\int_{\mathcal{S}} B_w\dot{\beta}_w ds = \int_{\mathcal{S}} B_w\dot{\kappa}'_1 ds = -\int_{\mathcal{S}} B'_w\dot{\kappa}_1 ds + \left[B_w\dot{\kappa}_1 \right]_A^B \qquad [8.52]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Moreover, by remembering the strain-rate-velocity relationships [7.17a,b], the VPP is transformed into[13]:

$$\begin{aligned}&\int_{\mathcal{S}}[\mathbf{t}\cdot(\mathbf{v}_C'-\boldsymbol{\omega}\times\mathbf{x}_C')+((M_t-B_w'-q_w)\,\mathbf{a}_1\\&+(\mathbf{m}_\pi-\mathbf{r}_C\times N\mathbf{a}_1))\cdot\boldsymbol{\omega}']ds\\&=\int_{\mathcal{S}}(\mathbf{p}\cdot\mathbf{v}_C+\mathbf{c}\cdot\boldsymbol{\omega})\,ds\\&+\sum_{H=A}^{B}(\mathbf{P}_H\cdot\mathbf{v}_{CH}+\mathbf{C}_H\cdot\boldsymbol{\omega}_H+Q_{wH}\mathbf{a}_1\cdot\boldsymbol{\omega}_H')-[B_w\mathbf{a}_1\cdot\boldsymbol{\omega}']_A^B\\&\forall\,(\mathbf{v}_C,\boldsymbol{\omega})\end{aligned}\qquad [8.53]$$

After another integration by parts, we get:

$$\begin{aligned}&\int_{\mathcal{S}}\big[(\mathbf{t}'+\mathbf{p})\cdot\mathbf{v}_C+(((M_t-B_w'-q_w)\,\mathbf{a}_1)'\\&+(\mathbf{m}_\pi-\mathbf{r}_C\times\mathbf{a}_1N)'+\mathbf{x}_C'\times\mathbf{t}+\mathbf{c})\cdot\boldsymbol{\omega}\big]ds\\&+\sum_{H=A}^{B}[(\mathbf{P}_H\pm\mathbf{t}_H)\cdot\mathbf{v}_{CH}+(\mathbf{C}_H\pm(M_{tH}-B_{wH}'-q_{wH})\,\mathbf{a}_1\\&\pm(\mathbf{m}_{\pi H}-\mathbf{r}_C\times N_H\mathbf{a}_1))\cdot\boldsymbol{\omega}_H]\\&+(Q_{wH}\pm B_{wH})\,\mathbf{a}_1\cdot\boldsymbol{\omega}_H'=0,\qquad\forall\,(\mathbf{v}_C,\boldsymbol{\omega})\end{aligned}\qquad [8.54]$$

Finally, we obtain the field equations:

$$\begin{aligned}&\mathbf{t}'+\mathbf{p}=\mathbf{0}\\&((M_t-B_w'-q_w)\,\mathbf{a}_1)'+(\mathbf{m}_\pi-\mathbf{r}_C\times N\mathbf{a}_1)'+\mathbf{x}_C'\times\mathbf{t}+\mathbf{c}=\mathbf{0}\end{aligned}\qquad [8.55]$$

and the boundary conditions:

$$\begin{aligned}&(\mathbf{P}_H\pm\mathbf{t}_H)\cdot\mathbf{v}_{CH}=\mathbf{0}\\&[\mathbf{C}\pm(M_t-B_w'-q_w)\,\mathbf{a}_1\pm(\mathbf{m}_\pi-\mathbf{r}_C\times N\mathbf{a}_1)]_H\cdot\boldsymbol{\omega}_H=\mathbf{0}\\&(Q_w\pm B_w)_H\,\mathbf{a}_1\cdot\boldsymbol{\omega}_H'=0\end{aligned}\qquad [8.56]$$

where the index π, as usual, denotes, the component in the $(\mathbf{a}_1,\mathbf{a}_2)$-plane.

13. As a matter of fact, $\mathbf{v}_C'-\boldsymbol{\omega}\times\mathbf{x}_C'=\mathbf{R}\dot{\mathbf{e}}_C=\mathbf{R}\,(\dot{\varepsilon}_C\bar{\mathbf{a}}_1+\dot{\boldsymbol{\gamma}}_C)$ and $\boldsymbol{\omega}'=\mathbf{R}\dot{\mathbf{k}}=\mathbf{R}\left(\dot{\kappa}_1\bar{\mathbf{a}}_1+\dot{\mathbf{k}}_\pi\right)$; moreover, from the latter, $\dot{\kappa}_1=\boldsymbol{\omega}'\cdot\mathbf{a}_1$.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

REMARK 8.7. If we compare equations [8.55] and [8.56] with equations [7.26] and [7.27], relevant to the distortion-unconstrained beam, we note that: (a) the balance equation relevant to the distortional-variable disappears; (b) the torsional moment M_1 is modified by the warping stresses D_w, and, moreover, it combines with $-B_w'$ in the equilibrium equation. The non-uniform torsion case, of section 8.3, is derived as a particular case.

The elastic law

The elastic law must link the active stresses, N, M_t, M_2, M_3 to the admissible strains $\varepsilon_G, \kappa_1, \kappa_2, \kappa_3$. It is derived by the identified law [7.66] for distortion-unconstrained TWB, by substituting the constraint [8.1] and adding to each other the rows relevant to M_1 and D_w. The following relationships are found:

$$\begin{pmatrix} N \\ M_t \\ M_2 \\ M_3 \\ B_w \end{pmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 \\ 0 & GJ_t & 0 & 0 & 0 \\ 0 & 0 & EJ_2 & 0 & 0 \\ 0 & 0 & 0 & EJ_3 & 0 \\ 0 & 0 & 0 & 0 & E\mathit{\Gamma}_w \end{bmatrix} \begin{pmatrix} \varepsilon_G \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \beta_w \end{pmatrix} + \boldsymbol{f}(\boldsymbol{\varepsilon}) \qquad [8.57]$$

where J_t is defined in equation (8.42), and, moreover:

$$\boldsymbol{f}(\boldsymbol{\varepsilon}) := \begin{pmatrix} \frac{1}{2} EJ_C \kappa_1^2 \\ \kappa_1 \left[EJ_C \varepsilon_G - EY_{3C}\kappa_3 + EY_{2C}\kappa_2 + E\mathit{\Gamma}_{wC}\beta_w \right] \\ \kappa_1^2 \left[\frac{1}{2} EY_{2C} + GY_{n2} + GY_{w2} \right] \\ \kappa_1^2 \left[-\frac{1}{2} EY_{3C} + GY_{n3} + GY_{w3} \right] \\ \frac{1}{2} E\mathit{\Gamma}_{wC}\kappa_1^2 \end{pmatrix} + \begin{pmatrix} 0 \\ G\kappa_1 \left[\mathbf{k}_\pi \cdot \mathbf{y}_n + G\mathbf{k}_\pi \cdot \mathbf{y}_w + G\mathbf{k}_\pi \cdot \mathbf{\Gamma}_w \mathbf{k}_\pi \right] \\ \kappa_1^2 \left[G\mathit{\Gamma}_{w22}\kappa_2 + G\mathit{\Gamma}_{w23}\kappa_3 \right] \\ \kappa_1^2 \left[G\mathit{\Gamma}_{w32}\kappa_2 + G\mathit{\Gamma}_{w33}\kappa_3 \right] \\ 0 \end{pmatrix} \qquad [8.58]$$

The Fundamental Problem

The Fundamental Problem for the unshearable, warpable, distortion-constrained TWB, is governed by the following equations.

1) Four strain–displacement relationships [7.16a,d,e,f];

2) Two unshearability conditions, derived from equations [7.16b,c] by zeroing γ_{2C}, γ_{3C};

3) One relationship, $\beta_w = \kappa_1'$, consequent to the distortion constraint;

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

4) Six balance equations [8.55];
5) Five elastic laws [8.57].

The mechanical boundary conditions are given by equations [8.56], which must be supplemented by proper geometric conditions.

The previous equations express the hybrid formulation for the TWB under analysis. They are a system of 18 equations in the following unknowns: (a) the six displacements $u_{1C}, u_{2C}, u_{3C}, \theta_1, \theta_2, \theta_3$; (b) the five admissible strains $\varepsilon_C, \kappa_1, \kappa_2, \kappa_3, \beta_w$; (c) the five active stresses N, M_t, M_2, M_3, B_w and the two reactive shear-stresses T_2, T_3.

8.5 Cross-deformable planar TWB

Active and reactive stresses

Because of the *nonlinear* Brazier constraint [8.2], the *linear* constraint $\dot{\alpha}_\pi = 2\kappa\dot{\kappa}$ holds for the strain-rates. Therefore, when a Lagrange multiplier R is enclosed in the expression [7.85] of internal virtual power (already accounting for unshearability), the latter becomes:

$$\mathcal{P}_{int} = \int_{\mathcal{S}} \left(N_a \dot{\varepsilon}_G + M_a \dot{\kappa} + D_{\pi a} \dot{\alpha}_\pi + B_{\pi a} \dot{\beta}_\pi \right) ds + \int_{\mathcal{S}} R \left(\dot{\alpha}_\pi - 2\kappa\dot{\kappa} \right) ds \quad [8.59]$$

or:

$$\mathcal{P}_{int} = \int_{\mathcal{S}} \left[N_a \dot{\varepsilon}_G + \left(M_a - 2\kappa R \right) \dot{\kappa} + \left(D_{\pi a} + R \right) \dot{\alpha}_\pi + B_{\pi a} \dot{\beta}_\pi \right] ds \quad [8.60]$$

It follows that the total stresses, conjugate of the strain-rates, are made of an active and a reactive parts:

$$\begin{aligned} N &:= N_a \\ M &:= M_a - 2\kappa R \\ D_\pi &:= D_{\pi a} + R \\ B_\pi &:= B_{\pi a} \end{aligned} \quad [8.61]$$

Note that, due to the nonlinearity, the reactive part of the stresses explicitly depends on the configuration. If the reactive force R is condensed, a *purely active stress* $M_f := M + 2D_\pi \kappa = M_a + 2D_{\pi a}\kappa$ is found, conjugate of the flexure-rate.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The same result, of course, can be achieved if the strain-rate constraint $\dot{\alpha}_\pi = 2\kappa\dot{\kappa}$ is directly substituted in the internal virtual power [7.85], thus renouncing the evaluation of the reactive stress R. Indeed, we have:

$$\mathcal{P}_{int} := \int_{\mathcal{S}} \left(N\dot{\varepsilon}_G + (M + 2D_\pi\kappa)\,\dot{\kappa} + B_\pi\dot{\beta}_\pi \right) ds \qquad [8.62]$$

from which it appears that $M_f := M + 2D_\pi\kappa$ is the purely active stress conjugate of the flexural curvature rate.

The balance equations for the constrained problem

We derive the balance equations for the constrained TWB by the VPP. We follow the displacement approach to account for the Brazier constraint; thus, we take the expression [8.62] for the internal power and the following expression for the external power:

$$\begin{aligned}\mathcal{P}_{ext} :=& \int_{\mathcal{S}} \left(\mathbf{p}\cdot\mathbf{v}_C + c\omega + 2q_\pi\kappa\dot{\kappa}\right) ds \\ &+ \sum_{H=A}^{B} \left(\mathbf{P}_H\cdot\mathbf{v}_{CH} + C_H\omega_H + 2Q_{\pi H}\kappa_H\dot{\kappa}_H\right)\end{aligned} \qquad [8.63]$$

in which we used $a_\pi = \kappa^2$, and therefore $\dot{a}_\pi = 2\kappa\dot{\kappa}$. Moreover, we account for the unshearability by adding the zero-terms $T\dot{\gamma}_C$ to the internal power, where T is a new scalar Lagrangian parameter. In conclusion, by remembering (equation [7.21a]) that $\dot{\varepsilon}_G = \dot{\varepsilon}_C + \dot{\kappa}r_{2C}$, the VPP becomes:

$$\begin{aligned}&\int_{\mathcal{S}} \left(N\dot{\varepsilon}_C + T\dot{\gamma}_C + (M_f + Nr_{2C} - 2\kappa q_\pi)\,\dot{\kappa} + B_\pi\dot{\beta}_\pi \right) ds \\ &= \int_{\mathcal{S}} (\mathbf{p}\cdot\mathbf{v}_C + c\omega)\, ds + \sum_{H=A}^{B} \left(\mathbf{P}_H\cdot\mathbf{v}_{CH} + C_H\omega_H + 2\kappa_H Q_{\pi H}\dot{\kappa}_H\right)\end{aligned} \qquad [8.64]$$

Since $\dot{\beta}_\pi = \dot{\alpha}'_\pi$, $\dot{\alpha}_\pi = 2\kappa\dot{\kappa}$, an integration by parts provides:

$$\int_{\mathcal{S}} B_\pi\dot{\beta}_\pi ds = -\int_{\mathcal{S}} B'_\pi\dot{\alpha}_\pi ds + \left[B_\pi\dot{\alpha}_\pi\right]_A^B = -2\int_{\mathcal{S}} B'_\pi\kappa\dot{\kappa} ds + 2\left[B_\pi\kappa\dot{\kappa}\right]_A^B \qquad [8.65]$$

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Because of $N\dot{\varepsilon}_C + T\dot{\gamma}_C = \mathbf{t}\cdot\mathbf{R}\dot{\mathbf{e}}_C = \mathbf{t}\cdot(\mathbf{v}'_C - \boldsymbol{\omega}\times\mathbf{x}'_C)$, where $\boldsymbol{\omega} = \omega\mathbf{a}_3$, the VPP assumes the form:

$$\begin{aligned}
&\int_{\mathcal{S}} \left[\mathbf{t}\cdot(\mathbf{v}'_C - \boldsymbol{\omega}\times\mathbf{x}'_C) + (M_f + Nr_{2C} - 2\,(B'_\pi + q_\pi)\,\kappa)\,\dot{\kappa}\right] ds \\
&\quad = \int_{\mathcal{S}} (\mathbf{p}\cdot\mathbf{v}_C + c\omega)\, ds \\
&\quad + \sum_{H=A}^{B} (\mathbf{P}_H\cdot\mathbf{v}_{CH} + C_H\omega_H + 2\kappa_H Q_{\pi H}\dot{\kappa}_H) - 2\,[B_\pi\kappa\dot{\kappa}]_A^B
\end{aligned} \tag{8.66}$$

By integrating by parts, and accounting for $\boldsymbol{\omega} = \omega\mathbf{a}_3 = \dot{\theta}\mathbf{a}_3$, $\kappa = \theta'$, and $\dot{\kappa} = \omega'$, we finally obtain:

$$\begin{aligned}
&\int_{\mathcal{S}} \Big[(\mathbf{t}' + \mathbf{p})\cdot\mathbf{v}_C \\
&\quad + \Big((M_f + Nr_{2C} - 2\,(B'_\pi + q_\pi)\,\kappa)' + \mathbf{x}'_C\times\mathbf{t}\cdot\mathbf{a}_3 + c\Big)\,\omega\Big] ds \\
&\quad + \sum_{H=A}^{B} \big[(\mathbf{P}_H \pm \mathbf{t}_H)\cdot\mathbf{v}_{CH} \\
&\quad + (C_H \pm (M_f + Nr_{2C}) \mp 2\,(B'_\pi + q_\pi)\,\kappa)\,\omega_H\big] \\
&\quad - 2\kappa_H\,(Q_{\pi H} \pm B_{\pi H})\,\omega'_H = 0, \qquad \forall\,(\mathbf{v}_C, \omega)
\end{aligned} \tag{8.67}$$

This supplies the balance equations:

$$\begin{aligned}
&\mathbf{t}' + \mathbf{p} = 0 \\
&(M_f + Nr_{2C} - 2\,(B'_\pi + q_\pi)\,\kappa)' + \mathbf{x}'_C\times\mathbf{t}\cdot\mathbf{a}_3 + c = 0
\end{aligned} \tag{8.68}$$

with the boundary conditions:

$$\begin{aligned}
&[\mathbf{P} \pm \mathbf{t}]_H\cdot\mathbf{v}_{CH} = \mathbf{0} \\
&[C \pm (M_f + Nr_{2C} - 2\,(B'_\pi + q_\pi)\,\kappa)]_H\,\omega_H = 0
\end{aligned} \tag{8.69}$$

REMARK 8.8. Coupling between distortional and rigid-model stresses only appear in the nonlinear part of the balance equations.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The Fundamental Problem

The Fundamental Problem for the 1D, planar, unshearable, distortion-constrained TWB is governed by the following equations (compare them with those for the distortion-free problem, section 7.7.3).

1) The strain–displacement relationships:

$$\begin{aligned}&\varepsilon_C = (1 + u'_{1C}) \cos\theta + u'_{2C} \sin\theta - 1\\&\kappa = \theta'\\&\beta_\pi = 2\theta'\theta''\end{aligned} \quad [8.70]$$

2) The unshearability condition:

$$-(1 + u'_{1C}) \sin\theta + u'_{2C} \cos\theta = 0 \quad [8.71]$$

3) The balance equations [8.68]:

$$\begin{aligned}&N' - \kappa T + p_1 = 0\\&T' + \kappa N + p_2 = 0\\&\left(M_f + N r_{2C} - 2\left(B'_\pi + q_\pi\right)\kappa\right)' + (1 + \varepsilon_C)\, T + c = 0\end{aligned} \quad [8.72]$$

4) The elastic law [7.91], in which the constraint $\alpha_\pi = \kappa^2$ is substituted, and stresses are combined to eliminate the reactive part:

$$\begin{pmatrix} N \\ M_f \\ B_\pi \end{pmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EJ_3 & 0 \\ 0 & 0 & GJ_\kappa + GJ_\gamma \end{bmatrix} \begin{pmatrix} \varepsilon_G \\ \kappa \\ \beta_\pi \end{pmatrix} + \boldsymbol{f}(\boldsymbol{\varepsilon}) \quad [8.73]$$

where:

$$\boldsymbol{f}(\boldsymbol{\varepsilon}) := \begin{pmatrix} 0 \\ 2E\left(A_\chi + A_\varepsilon + 2J_{3\psi}\right)\kappa^3 + 3EJ_\psi \kappa^5 \\ 0 \end{pmatrix} \quad [8.74]$$

The problem is completed by the following mechanical boundary conditions:

$$\begin{aligned}&\left[P_1 \pm N\right]_H v_{1H} = 0\\&\left[P_2 \pm T\right]_H v_{2H} = 0\\&\left[C \pm \left(M_f + N r_{2C} - 2\left(B'_\pi + q_\pi\right)\kappa\right)\right]_H \omega_H = 0\end{aligned} \quad [8.75]$$

and proper geometric boundary conditions.

The previous equations constitute a system of 10 equations in the following unknowns: (a) the three displacements u_{1C}, u_{2C}, θ; (b) the three admissible strains $\varepsilon_C, \kappa, \beta_\pi$; (c) the three dual active stresses N, M_f, B_π and the reactive shear-stress T.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

8.6 Summary

In this chapter, we tackled *distortion-constrained* TWB. These are beams in which the distortional strains are not free, but which must satisfy some kinematic constraints. Thus, if we enforce the Vlasov condition, which requires the vanishing of the shear-strain on the middle-line of an open TWB, we obtain the *linear* constraint $\alpha_w = \kappa_1$; moreover, the warping function naturally springs as the sectorial area function of the linear theory. Similarly, if we enforce the Bredt condition, concerning the constancy of the shear-flow in a tubular section, we obtain the same constraint as before, and the related warping function. Finally, if we consider a symmetric TWB bent in its plane of symmetry, and enforce the Brazier constraint, which, on the grounds of (approximated) equilibrium conditions, relates the flattening of the cross-section to the bending curvature, we find the *nonlinear* constraint $\alpha_\pi = \kappa^2$ and the relevant distortional mode.

These kinds of constraints are more general than those considered throughout the book, which were confined to the vanishing of one or more strain components. To investigate their effect on the stresses, we first addressed, for educational purposes, the simple problem of non-uniform torsion of a warpable cross-undeformable beam. We observed that, by incorporating the constraint condition in the expression of the internal virtual power, we are able to distinguish an active and a reactive contribution: the first expresses the power spent by the stress that depends on the strain (irrespectively of the constitutive law); the second expresses the power spent by the contribution of the stress that is independent of the strain[14]. In some sense, the reactive virtual power can be seen as the power that the reactive stresses would spend in the strains *if they were allowed to violate* the constraint condition. When the internal power is written for the TWB in torsion, it is found that the torsional moment M_1 and the distortional moment D_w are both made up of an active and a reactive part, this latter being equal and opposite in sign, in order to satisfy the "Postulate of Perfect Constraint". The active part of D_w is found to vanish in open cross-section, but to be different from zero in tubular sections. The sum of the two total stresses is the *active torsional moment* $M_t := M_1 + D_w$ conjugate of the torsional curvature-rate. Most remarkably, *the kinematic constraint couples the balance equations* for locally rigid and locally deformable-beams, which are instead uncoupled for free distortional parameters. Moreover, this coupling manifests itself even in the linear part of the equations.

14. A useful example showing the double nature of the stress consists of an elastic truss-beam, to which one or more rigid trusses are added: the normal forces exerted by the elastic trusses are active stresses, the normal forces exerted by the rigid trusses are reactive stresses, which add themselves to the former.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The general problem of cross-underfomable, warpable TWB was successively addressed. The problem was formulated by direct elimination of the Vlasov/Bredt condition, but accounting for the unshearability of the beam via Lagrange multipliers. The previous consideration, valid for the simple case of torsion, was thus generalized.

As a last example, we introduced the Brazier constraint in a planar, unwarpable TWB. We observed that the flexural moment M and the in-plane distortional moment D_π are of an active-reactive nature. A generalized *active flexural moment* $M_f := M + 2D_\pi\kappa$ was defined, conjugate of the flexural curvature rate. Because of the nonlinearity of the constraint, it is explicitly configuration-dependent. The balance equations, derived as for the warpable beam, were found to be coupled only in their nonlinear part.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Bibliography

[ALI 03] ALIBERT J.-J., SEPPECHER P., DELL'ISOLA F., "Truss modular beams with deformation energy depending on higher displacement gradients", *Mathematics and Mechanics of Solids*, vol. 8, pp. 51–73, 2003.

[AMA 08] AMABILI M., *Nonlinear Vibrations and Stability of Shells and Plates*, Cambridge University Press, 2008.

[ANT 72] ANTMAN S.S., "The theory of rods", *Encyclopedia of Physics*, Springer-Verlag, 1972.

[ANT 05] ANTMAN S.S., *Nonlinear Problems of Elasticity*, Springer-Verlag, 2005.

[ARD 05] ARDEMA M.D., *Analytical Dynamics. Theory and Applications*, Kluwer Academic/Plenum Publishers, 2005.

[ARG 82] ARGYRIS J., "An excursion into large rotations", *Computer Methods in Applied Mechanics and Engineering*, vol. 32, pp. 85–155, 1982.

[ATA 97] ATANACKOVIĆ T.M., *Stability Theory of Elastic Rods*, World Scientific, 1997.

[AWR 12] AWREJCEWICZ J., *Classical Mechanics: Kinematics and Statics*, Springer-Verlag, 2012.

[BAR 10] BARBER J.R., *Elasticity*, Springer-Verlag, 2010.

[BAS 09] BASAGLIA C., CAMOTIM D., SILVESTRE N., "GBT-based local, distortional and global buckling analysis of thin-walled steel frames", *Thin-Walled Structures*, vol. 47, pp. 1246–1264, 2009.

[BAT 82] BATHE K.-J., *Finite Element Procedures*, Prentice Hall, 1982.

[BAŽ 03] BAŽANT Z. P., CEDOLIN L., *Stability of Structures*, Dover Publications, Inc., 2003.

[BEB 08] BEBIANO R., SILVESTRE N., CAMOTIM D., "Local and global vibration of thin-walled members subjected to compression and non-uniform bending", *Journal of Sound and Vibration*, vol. 315, pp. 509–535, 2008.

[BEN 99] BENDER C.M., ORSZAG S. A., *Advanced Mathematical Methods for Scientists and Engineers I*, Springer-Verlag, 1999.

[BEN 12] BENEDETTINI F., ALAGGIO R., ZULLI D., "Nonlinear coupling and instability in the forced dynamics of a non-shallow arch: theory and experiments", *Nonlinear Dynamics*, vol. 68, pp. 505–517, 2012.

Mathematical Models of Beams and Cables Angelo Luongo and Daniele Zulli

© 2013 ISTE Ltd. Published 2013 by ISTE Ltd.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[BER 09] BERDICHEVSKY V.L., *Variational Principles of Continuum Mechanics*, Springer-Verlag, 2009.

[BIG 12] BIGONI D., *Nonlinear Solid Mechanics*, Cambridge University Press, 2012.

[BIS 07] BISHOP R.H., *Mechatronic Systems, Sensors, and Actuators: Fundamentals and Modeling*, CRC Press, 2007.

[BLE 01] BLEVINS R., *Flow-Induced Vibration*, Krieger Publishing Company, Florida, 2001.

[BOL 63] BOLOTIN V.V., *Nonconservative Problems of the Theory of Elastic Stability*, Pergamon Press, 1963.

[BOL 64] BOLOTIN V.V., *The Dynamic Stability of Elastic Systems*, Holden-Day, Inc., 1964.

[BRA 27] BRAZIER L.G., "On the flexure of thin cylindrical shells and other 'thin' sections", *Proceedings of the Royal Society of London A*, vol. 116, pp. 104–114, 1927.

[BRA 02] BRANNON R., *Rotation: A Review of Useful Theorems Involving Proper Orthogonal Matrices Referenced to Three-Dimensional Space*, available at http://www.mech.utah.edu/ brannon/public/rotation.pdf, 2002.

[BRI 13] BRISKOW E., *Elementary Continuum Mechanics for Everyone*, Springer-Verlag, 2013.

[BUR 88] BURGESS J.J., TRIANTAFYLLOU M.S., "The elastic frequencies of cables", *Journal of Sound and Vibration*, vol. 120, no. 1, pp. 153–165, 1988.

[CAM 06] CAMOTIM D., SILVESTRE N., GONÇALVES R., DINIS P.B., "GBT-based structural analysis of thin-walled members: overview, recent progress and future developments", *Advances in Engineering Structures, Mechanics & Construction Solid Mechanics and Its Applications*, vol. 140, pp. 187–204, 2006.

[CAM 10] CAMOTIM D., BASAGLIA C., SILVESTRE N., "GBT buckling analysis of thin-walled steel frames: a state-of-the-art report", *Thin-Walled Structures*, vol. 48, pp. 726–743, 2010.

[CAP 89] CAPRIZ G., *Continua with Microstructure*, Springer-Verlag, 1989.

[CHA 11] CHALLAMEL N., "Higher-order shear beam theories and enriched continuum", *Mechanics Research Communications*, vol. 38, pp. 388–392, 2011.

[CHA 12] CHALLAMEL N., "On geometrically exact post-buckling of composite columns with interlayer slip – the partially composite elastica", *International Journal of Non-Linear Mechanics*, vol. 47, no. 3, pp. 7–17, 2012.

[CHA 13] CHAVES E.W.V., *Notes on Continuum Mechanics*, Springer-Verlag, 2013.

[CHO 01] CHOPRA A.K., *Dynamics of Structures*, Prentice Hall International, 2001.

[CIA 88] CIARLET P.G., *Mathematical Elasticity. Volume I: Three-Dimensional Elasticity*, Elsevier Science Publishers, 1988.

[CLO 03] CLOUGH R.W., PENZIEN J., *Dynamics of Structures*, 3rd ed., Computers & Structures, 2003.

[CON 07] CONSTANTINESCU A., KORSUNSKY A., *Elasticity with Mathematica*, Cambridge University Press, 2007.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[COU 53] COURANT R., HILBERT D., *Methods of Mathematical Physics. Volume I*, Interscience Publishers, 1953.

[CRE 78a] CRESPO DA SILVA M.R.M., GLYNN C.C., "Nonlinear flexural-flexural-torsional dynamics of inextensional beams. I. equations of motion", *Journal of Structural Mechanics*, vol. 6, no. 4, pp. 437–448, 1978.

[CRE 78b] CRESPO DA SILVA M.R.M., GLYNN C.C., "Nonlinear flexural-flexural-torsional dynamics of inextensional beams. II. forced motions", *Journal of Structural Mechanics.*, vol. 6, no. 4, pp. 449–461, 1978.

[CRE 88a] CRESPO DA SILVA M.R.M., "Non-linear flexural-flexural-torsional-extensional dynamics of beams-I. formulation", *International Journal of Solids and Structures*, vol. 24, no. 12, pp. 1225–1234, 1988.

[CRE 88b] CRESPO DA SILVA M.R.M., "Non-linear flexural-flexural-torsional-extensional dynamics of beams-II. response analysis", *International Journal of Solids and Structures*, vol. 24, no. 12, pp. 1235–1242, 1988.

[CRE 91] CRESPO DA SILVA M.R.M., "Equations of nonlinear analysis of 3D motions of beams", *Applied Mechanics Reviews*, vol. 44, no. 11, 1991.

[DEL 09] DELL'ISOLA F., SCIARRA G., VIDOLI S., "Generalized Hooke's law for isotropic second gradient materials", *Proceedings of the Royal Society of London A*, vol. 465, pp. 2177–2196, 2009.

[DEL 12] DELL'ISOLA F., PLACIDI L., "Variational principles are a powerful tool also for formulating field theories", *Variational Models and Methods in Solid and Fluid Mechanics*, Springer-Verlag, 2012.

[DEN 87] DEN HARTOG J.P., *Advanced Strength of Materials*, Dover Publications, Inc., 1987.

[DIC 90] DICARLO A., RIZZI N., TATONE A., "Continuum modelling of a beam-like latticed truss: identification of the constitutive functions for the contanct and inertial actions", *Meccanica*, vol. 25, pp. 168–174, 1990.

[DIC 96] DICARLO A., "A non-standard format for continuum mechanics", *Contemporary Research in the Mechanics and Mathematics of Materials*, CIMNE, Barcelona, 1996.

[DIC 99] DICARLO A., NARDINOCCHI P., "On the torsion of soft cylindrical shells", *Trends in Applications of Mathematics to Mechanics*, Chapman and Hall/CRC, 1999.

[DIC 01] DICARLO A., PODIO-GUIDUGLI P., WILLIAMS W.O., "Shell with thickness distension", *International Journal of Solids and Structures*, vol. 38, pp. 1201–1225, 2001.

[DI 03a] DI EGIDIO A., LUONGO A., VESTRONI F., "A nonlinear model for the dynamics of open cross-section thin-walled beams. Part 1: formulation", *International Journal of Non-Linear Mechanics*, vol. 38, no. 7, pp. 1067–1081, 2003.

[DI 03b] DI EGIDIO A., LUONGO A., VESTRONI F., "A nonlinear model for the dynamics of open cross-section thin-walled beams. Part 2: forced motion", *International Journal of Non-Linear Mechanics*, vol. 38, no. 7, pp. 1083–1094, 2003.

[DIM 11] DIMITRIENKO Y.I., *Nonlinear Continuum Mechanics and Large Inelastic Deformations*, Springer-Verlag, 2011.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[DYM 13] DYM C.L., SHAMES I. H., *Solid Mechanics: A Variational Approach*, Springer-Verlag, 2013.

[ELI 01] ELISHAKOFF I., LI Y., STARNES J.H., *Non-Classical Problems in the Theory of Elastic Stability*, Cambridge University Press, 2001.

[ERE 13] EREMEYEV V., LEBEDEV L., ALTENBACH H., *Foundations of Micropolar Mechanics*, Springer-Verlag, 2013.

[ESL 13] ESLAMI M.R., HETNARSKI R.B., IGNACZAK J., NODA N., SUMI N., TANIGAWA Y., *Theory of Elasticity and Thermal Stresses*, Springer-Verlag, 2013.

[FAB 09] FABIEN B.C., *Analytical System Dynamics: Modeling and Simulations*, Springer-Verlag, 2009.

[FER 06] FERTIS D.G., *Nonlinear Structural Engineering: With Unique Theories and Methods to Solve Effectively Complex Nonlinear Problems*, Springer-Verlag, 2006.

[FIN 08] FINN J.M., *Classical Mechanics*, Infinity Science Press LLC, 2008.

[FUN 01] FUNG Y.C., TONG P., *Classical and Computational Solid Mechanics*, World Scientific, 2001.

[GAL 07] GALLAVOTTI G., *The Elements of Mechanics*, Springer-Verlag, 1983.

[GAN 13] GANS R.F., *Engineering Dynamics: From the Lagrangian to Simulation*, Springer-Verlag, 2013.

[GAT 02] GATTI-BONO C., PERKINS N.C., "Dynamic analysis of loop formation in cables under compression", *International Journal of Offshore and Polar Engineering*, vol. 12, no. 3, 2002.

[GER 73] GERMAIN P., "The method of virtual power in continuum mechanics. Part 2: microstructure", *SIAM Journal on Applied Mathematics*, vol. 25, no. 3, pp. 556–575, 1973.

[GOL 80] GOLDSTEIN H., POOLE C., SAFKO J., *Classical Mechanics*, Addison Wesley, 1980.

[GON 07] GONÇALVES R., CAMOTIM D., "Thin-walled member plastic bifurcation analysis using generalised beam theory", *Advances in Engineering Software*, vol. 38, pp. 637–646, 2007.

[GON 10] GONÇALVES R., RITTO-CORRÊA M., CAMOTIM D., "A large displacement and finite rotation thin-walled beam formulation including cross-section deformation", *Computer Methods in Applied Mechanics and Engineering*, vol. 199, pp. 1627–1643, 2010.

[GOY 05] GOYAL S., PERKINS N.C., LEE C.L., "Nonlinear dynamics and loop formation in Kirchhoff Rods with implications to the mechanics of DNA and cables", *Journal of Computational Physics*, vol. 209, no. 1, pp. 371–389, 2005.

[GOY 07] GOYAL S., PERKINS N.C., "Nonlinear dynamic strand model with coupled tension and torsion", *arXiv preprint physics*, vol. 0702201, 2007.

[GRE 92] GREEN A.E., ZERNA W., *Theoretical Elasticity*, Dover Publications, Inc., 1992.

[GRE 10] GREINER W., *Classical Mechanics: Systems of Particles and Hamiltonian Dynamics*, Springer-Verlag, 2010.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[GUR 72] GURTIN M.E., "The linear theory of elasticity", *Encyclopedia of Physics*, Springer-Verlag, 1972.

[GUR 82] GURTIN M.E., *An Introduction to Continuum Mechanics*, Academic Press, 1982.

[GUR 83] GURTIN M.E., *Topics in Finite Elasticity*, Society for Industrial and Applied Mechanics, 1983.

[GUR 00] GURTIN M.E., *Configurational Forces as Basic Concepts of Continuum Physics*, Springer-Verlag, 2000.

[GUR 10] GURTIN M.E., FRIED E., ANAND L., *The Mechanics and Thermodynamics of Continua*, Cambridge University Press, 2010.

[HAR 13] HARTMANN F., *Greens Functions and Finite Elements*, Springer-Verlag, 2013.

[HOD 06] HODGES D.H., *Nonlinear Composite Beam Theory*, American Institute of Aeronautics and Astronautics, 2006.

[HOD 11] HODGES D.H., PIERCE G.A., *Introduction to Structural Dynamics and Aeroelasticity*, Cambridge University Press, 2011.

[HOL 00] HOLZAPFEL G.A., *Nonlinear Solid Mechanics*, John Wiley & Sons, 2000.

[HOW 09] HOWELL R., KOZYREFF G., OCKENDON J., *Applied Solid Mechanics*, Cambridge University Press, 2009.

[IBR 04] IBRAHIM R.A., "Nonlinear vibrations of suspended cables-Part III: random excitation and interaction with fluid flow", *Applied Mechanics Reviews*, vol. 57, no. 6, pp. 515–549, 2004.

[IBR 09] IBRAHIMBEGOVICH A., *Nonlinear Solid Mechanics: Theoretical Formulations and Finite Element Solution Methods*, Springer-Verlag, 2009.

[IRV 74] IRVINE H.M., CAUGHEY T.H., "The linear theory of free vibrations of a suspended cable", *Proceedingd of the Royal Society of London A*, vol. 341, pp. 299–315, 1974.

[IRV 81] IRVINE H.M., *Cable Structures*, MIT Press, 1981.

[KOK 06] KOKS D., *Explorations in Mathematical Physics*, Springer Science+Business Media, LLC, 2006.

[KUI 99] KUIPERS J.B., *Quaternions and Rotation Sequences*, Princeton University Press, 1999.

[LAC 13] LACARBONARA W., *Nonlinear Structural Mechanics: Theory, Dynamical Phenomena and Modeling*, Springer-Verlag, 2013.

[LAN 70] LANDAU L.D., LIFSHITZ E.M., *Theory of Elasticity*, Pergamon Press, 1970.

[LAN 97] LANCZOS C., *Linear Differential Operators*, Dover Publications, Inc., 1997.

[LEE 92] LEE C.L., PERKINS N.C., "Nonlinear oscillations of suspended cables containing a two-to-one internal resonance", *Nonlinear Dynamics*, vol. 3, pp. 465–490, 1992.

[LEI 74] LEIPHOLZ H., *Theory of Elasticity*, Noordhoff International Publishing, 1974.

[LEI 87] LEIPHOLZ H., *Stability Theory: An Introduction to the Stability of Dynamic Systems and Rigid Bodies*, John Wiley & Sons, 1987.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[LIB 06] LIBRESCU L., SONG O., *Thin-Walled Composite Beams: Theory and Applications.* Springer-Verlag, 2006.

[LOF 13] LOFRANO E., PAOLONE A., RUTA G.C., "A numerical approach for the stability analysis of open thin-walled beams", *Mechanics Research Communications*, vol. 48, pp. 76–86, 2013.

[LOV 44] LOVE A.E.H., *A Treatise on the Mathematical Theory of Elasticity*, Dover Publications, 1944.

[LOV 89] LOVELOCK D., HANNO R., *Tensors, Differential Forms, and Variational Principles*, Dover Publications, 1989.

[LU 94] LU C.L., PERKINS N.C., "Nonlinear spatial equilibria and stability of cables under uni-axial torque and thrus", *Journal of Applied Mechanics*, vol. 61, pp. 879–886, 1994.

[LUO 86] LUONGO A., REGA G., VESTRONI F., "On nonlinear dynamics of planar shear indeformable beams", *Journal of Applied Mechanics*, vol. 53, no. 3, pp. 619–624, 1986.

[LUO 98] LUONGO A., PICCARDO G., "Non-linear galloping of sagged cables in 1:2 internal resonance", *Journal of Sound and Vibrations*, vol. 214, no. 5, pp. 915–940, 1998.

[LUO 07] LUONGO A., ZULLI D., PICCARDO G., "A linear curved-beam model for the analysis of galloping in suspended cables", *Journal of Mechanics of Materials and Structures*, vol. 2, no. 4, pp. 675–694, 2007.

[LUO 08] LUONGO A., ZULLI D., PICCARDO G., "Analytical and numerical approaches to nonlinear galloping of internally resonant suspended cables", *Journal of Sound and Vibrations*, vol. 315, no. 3, pp. 375–393, 2008.

[LUO 09] LUONGO A., ZULLI D., PICCARDO G., "On the effect of twist angle on nonlinear galloping of suspended cables", *Computers & Structures*, vol. 87, pp. 1003–1014, 2009.

[LUO 12] LUONGO A., ZULLI D., "Dynamic instability of inclined cables under combined wind flow and support motion", *Nonlinear Dynamics*, vol. 67, no. 1, pp. 71–87, 2012.

[LUR 05] LURIE A.I., BELYAEV A., *Theory of Elasticity*, Springer-Verlag, 2005.

[MAG 12] MAGRAB E.B., *Vibrations of Elastic Systems*, Springer-Verlag, 2012.

[MAL 69] MALVERN L.E., *Introduction to the Mechanics of a Continuous Medium*, Prentice Hall, 1969.

[MAR 93] MARSDEN J.E., HUGHES T.J.R., *Mathematical Foundations of Elasticity*, Dover Publications, 1993.

[MAR 12] MARGHITU D.B., DUPAC M., *Advanced Dynamics*, Springer-Verlag, 2012.

[MEI 70] MEIROVITCH L., *Methods of Analytical Dynamics*, McGraw-Hill, 1970.

[MEI 80] MEIROVITCH L., *Computational Methods in Structural Dynamics*, Sijthoff & Noordhoff, 1980.

[MEI 97] MEIROVITCH L., *Principles and Techniques of Vibrations*, Prentice Hall International, 1997.

[MEI 01] MEIROVITCH L., *Fundamentals of Vibrations*, McGraw-Hill, 2001.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[MUR 86] MURRAY N.W., *Introduction to the Theory of Thin-Walled Structures*, Clarendon Press, 1986.

[MUR 00] MURDOCH A.I., "On objectivity and material symmetry for simple elastic solids", *Journal of Elasticity*, vol. 60, pp. 233–242, 2000.

[NAY 73] NAYFEH A.H., *Perturbation Methods*, John Wiley & Sons, 1973.

[NAY 79] NAYFEH A.H., MOOK D.T., *Nonlinear Oscillations*, John Wiley & Sons, 1979.

[NAY 04] NAYFEH A.H., PAI P.F., *Linear and Nonlinear Structural Mechanics*, John Wiley & Sons, 2004.

[OBO 13] OBODAN N.I., LEBEDEYEV O.G., GROMOV V.A., *Nonlinear Behaviour and Stability of Thin-Walled Shells*, Springer-Verlag, 2013.

[ODE 82] ODEN J.T., REDDY J.N., *Variational Methods in Theoretical Mechanics*, Springer-Verlag, 1982.

[OGD 97] OGDEN R.W., *Non-Linear Elastic Deformations*, Dover Publications, 1997.

[OGD 07] OGDEN R.W., "Incremental statics and dynamics of pre-stressed elastic materials", *Waves in Nonlinear Pre-Stressed Materials*, Springer-Verlag, 2007.

[PER 87] PERKINS N.C., MOTE C.D.JR., "Three dimensional vibration of traveling elastic cables", *Journal of Sound and Vibration*, vol. 114, no. 2, pp. 325–340, 1987.

[PIG 92] PIGNATARO M., RIZZI N., LUONGO A., *Stability, Bifurcation and Postcritical Behaviour of Elastic Structures*, Elsevier Science Ltd., 1992.

[PIG 09] PIGNATARO M., RIZZI N.L., RUTA G.C., VARANO V., "The effects of warping constraints on the buckling of thin-walled structures", *Journal of Mechanics of Materials and Structures*, vol. 4, pp. 1711–1727, 2009.

[POD 00] PODIO-GUIDUGLI P., "A primer in elasticity", *Journal of Elasticity*, vol. 58, no. 1, pp. 1–104, 2000.

[PRE 13] PREUMONT A., *Twelve Lectures on Structural Dynamics*, Springer-Verlag, 2013.

[RED 02] REDDY J.N., *Energy Principles and Variational Methods in Applied Mechanics*, John Wiley & Sons, 2002.

[RED 10] REDDY J.N., *Principles of Continuum Mechanics: A Study of Conservation Principles with Applications*, Cambridge University Press, 2010.

[RED 13] REDDY J.N., *An Introduction to Continuum Mechanics with Applications*, Cambridge University Press, 2013.

[REG 04a] REGA G., "Nonlinear vibrations of suspended cables-Part I: modeling and analysis", *Applied Mechanics Reviews*, vol. 57, no. 6, pp. 443–478, 2004.

[REG 04b] REGA G., "Nonlinear vibrations of suspended cables-Part II: deterministic phenomena", *Applied Mechanics Reviews*, vol. 57, no. 6, pp. 479–514, 2004.

[REI 59] REISSNER E., "On torsion of thin cylindrical shells", *Journal of the Mechanics and Physics of Solids*, vol. 7, pp. 157–162, 1959.

[REI 83a] REISSNER E., "On a simple variational analysis of small finite deformations of prismatical beams", *ZAMP* vol. 34, pp. 642–647, 1983.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[REI 83b] REISSNER E., "On some problems of buckling of prismatical beams under the influence of axial and transverse loads", *ZAMP* vol. 34, pp. 649–667, 1983.

[REI 84] REISSNER E., "On a variational analysis of finite deformations of prismatical beams and on the effect of warping stiffness on buckling loads", *ZAMP* vol. 35, pp. 247–251, 1984.

[REI 87] REISSNER E., REISSNER, J.E., WAN, F.Y.M., "On lateral buckling of end-loaded cantilevers, including the effect of warping stiffness", *Computational Mechanics*, vol. 2, pp. 137–147, 1987.

[RIZ 96] RIZZI N., TATONE A., "Nonstandard models for thin-walled beams with a view to applications", *Journal of Applied Mechanics*, vol. 63, pp. 399–403, 1996.

[ROM 06] ROMANO A., LANCELLOTTA R., MARASCO A., *Continuum Mechanics Using Mathematica*, Birkhäuser, 2006.

[ROS 11] ROSSIKHIN Y.A., SHITIKOVA M.V., *Dynamic Response of Pre-Stressed Spatially Curved Thin-Walled Beams of Open Profile*, Springer-Verlag, 2011.

[RUB 97] RUBIN M.B., "An intrinsic formulation for nonlinear elastic rods", *International Journal of Solids and Structures*, vol. 34, pp. 4191–4212, 1997.

[RUB 00] RUBIN M.B., *Cosserat Theories: Shells, Rods and Points. Solid Mechanics and its Applications*, Kluwer Academic Publishers, 2000.

[RUT 06] RUTA G.C., PIGNATARO M., RIZZI N.L., "A direct one-dimensional beam model for the flexural-torsional buckling of thin-walled beams", *Journal of Mechanics of Materials and Structures*, vol. 1, no. 8, pp. 1479–1496, 2006.

[SEP 11] SEPPECHER P., ALIBERT J.-J., DELL'ISOLA F., "Linear elastic trusses leading to continua with exotic mechanical interactions", *Journal of Physics: Conference Series*, vol. 319, pp. 1–13, 2011.

[SIL 03] SILVESTRE N., CAMOTIM D., "Nonlinear generalized beam theory for cold-formed steel members", *International Journal of Structural Stability and Dynamics*, vol. 3, pp. 461–490, 2003.

[SIL 10] SILVESTRE N., CAMOTIM D., "On the mechanics of distortion in thin-walled open sections", *Thin-Walled Structures*, vol. 48, pp. 469–481, 2010.

[SIM 85] SIMO J.C., "A finite strain beam formulation. The three-dimensional dynamic problem. Part I", *Computer Methods in Applied Mechanics and Engineering*, vol. 49, pp. 55–70, 1985.

[SIM 86] SIMO J.C., VU-QUOC L., "A three-dimensional finite strain rod model. Part II: Computational aspects", *Computer Methods in Applied Mechanics and Engineering*, vol. 58, pp. 79–116, 1986.

[SIM 88] SIMO J.C., VU-QUOC L., "On the dynamics in space of rods undergoing large motions – a geometrically exact approach", *Computer Methods in Applied Mechanics and Engineering*, vol. 66, pp. 125–161, 1988.

[SIM 91] SIMO J.C., VU-QUOC L., "A geometrically-exact rod model incorporating shear and torsion-warping deformation", *International Journal of Solids and Structures*, vol. 27, no. 3, pp. 371–393, 1991.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[SIM 06] SIMITES G.J., HODGES D.H., *Fundamentals of Structural Stability*, Elsevier 2006.

[SLA 99] SLABAUGH G., *Computing Euler Angles from a Rotation Matrix*, available at http://www.soi.city.ac.uk/~sbbh653/publications/euler.pdf, 1999.

[SOU 73] SOUTAS-LITTLE R.W., *Elasticity*, Dover Publications, 1973.

[TIM 51] TIMOSHENKO S.P., GOODIER J.N., *Theory of Elasticity*, McGraw-Hill, 1951.

[TIM 59] TIMOSHENKO S.P., WOINOWSKY-KRIEGER S., *Theory of Plates and Shells*, McGraw-Hill, 1959.

[TIM 63] TIMOSHENKO S.P., GERE J.M., *Theory of Elastic Stability*, McGraw-Hill, 1963.

[TIM 65] TIMOSHENKO S.P., YOUNG D.H., *Theory of Structures*, McGraw-Hill, 1965.

[TJA 98] TJAVARAS A.A., ZHU Q., LIU Y., TRIANTAFYLLOU M.S., YUE D.K.P., “The mechanics of highly-extensible cables”, *Journal of Sound and Vibration*, vol. 213, no. 4, pp. 709–737, 1998.

[TRI 84] TRIANTAFYLLOU M.S., “The dynamics of taut inclined cables”, *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 37, no. 3, pp. 421–440, 1984.

[TRU 66] TRUESDELL C., *The Elements of Continuum Mechanics*, Springer-Verlag, 1966.

[TRU 77] TRUESDELL C., *A First Course in Rational Continuum Mechanics*, Academic Press, 1977.

[TRU 04] TRUESDELL C., NOLL W., *The Non-Linear Field Theories of Mechanics*, Springer-Verlag, 2004.

[VIL 77] VILLAGGIO P., *Qualitative Methods in Elasticity*, Noordhoff International Publishing, 1977.

[VIL 97] VILLAGGIO P., *Mathematical Models for Elastic Structures*, Cambridge University Press, 1997.

[VIN 89] VINSON J.R., *The Behavior of Thin Walled Structures: Beams, Plates and Shells*, Kluwer Academic Publishers, 1989.

[VLA 61] VLASOV V.Z., *Thin-Walled Elastic Beams*, National Science Foundation and Department of Commerce, 1961.

[VOR 99] VOROVICH I.I., *Nonlinear Theory of Shallow Shells*, Springer-Verlag, 1999.

[WAN 00] WANG C.M., REDDY J.N., LEE K.H., *Shear Deformable Beams and Plates*, Elsevier, 2000.

[WAS 82] WASHIZU K., *Variational Methods in Elasticity and Plasticity*, Pergamon Press, 1982.

[WEG 09] WEGGNER J.L., HADDOW J.B., *Elements of Continuum Mechanics and Thermodynamics*, Cambridge University Press, 2009.

[WEI 74] WEINSTOCK R., *Calculus of Variations*, Dover Publications, 1974.

[WRI 08] WRIGGERS P., *Nonlinear Finite Element Methods*, Springer-Verlag, 2008.

[YOK 01] YOKOTA J.W., BEKELE S.A., STEIGMANN D.J., “Simulating the nonlinear dynamics of an elastic cable”, *AIAA Journal*, vol. 39, no. 3, pp. 504–510, 2001.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

[YU 05] YU W., HODGES D.H., VOLOVOI V.V., FUCHS E.D., "A generalized Vlasov theory for composite beams", *Thin-Walled Structures*, vol. 43, no. 9, pp. 1493–1511, 2005.

[ZAR 94] ZARETZKY C.L., CRESPO DA SILVA M.R.M., "Experimental investigation of non-linear modal coupling in the response of cantilever beams", *Journal of Sound and Vibration*, vol. 174, no. 2, pp. 145–167, 1994.

[ZIE 05] ZIENKIEWICZ O.C., TAYLOR R.L., *The Finite Element Method for Solid and Structural Mechanics*, Butterworth-Heinemann, 2005.

[ZUL 09] ZULLI D., ALAGGIO R., BENEDETTINI F., "Non-linear dynamics of curved beams. Part 1: Formulation", *International Journal of Non-Linear Mechanics*, vol. 44, pp. 623–629, 2009.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Index

Mathematical Models of Beams and Cables Angelo Luongo and Daniele Zulli

© 2013 ISTE Ltd. Published 2013 by ISTE Ltd.

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

P

R

S

T

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Downloaded from https://onlinelibrary.wiley.com/doi/ by Dhaka University of Engineerin, Wiley Online Library on [19/12/2023]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License